

INTEGRAL STABILITY OF CALDERÓN INVERSE CONDUCTIVITY PROBLEM IN THE PLANE

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Abstract

It is proved that, in two dimensions, the Calderón inverse conductivity problem in Lipschitz domains is stable in the L^p sense when the conductivities are uniformly bounded in any fractional Sobolev space $W^{\alpha,p}$ $\alpha > 0, 1 < p < \infty$.

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1 Introduction

Calderón inverse problem consists in the determination of an isotropic L^∞ conductivity coefficient γ on Ω from boundary measurements. These measurements are given by the Dirichlet to Neumann map Λ_γ , defined for a function f on $\partial\Omega$ as the Neumann value

$$\Lambda_\gamma(f) = \gamma \frac{\partial}{\partial \nu} u,$$

where u is the solution of the Dirichlet boundary value problem

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 \\ u|_{\partial\Omega} = f \end{cases} \quad (1.1)$$

and $\frac{\partial}{\partial \nu}$ denotes the outer normal derivative. For general domain and conductivities where the pointwise definition $\gamma \frac{\partial}{\partial \nu} u$ has no meaning, the Dirichlet to Neumann map

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega) \quad (1.2)$$

can be defined by

$$\langle \Lambda_\gamma(f), \varphi_0 \rangle = \int_{\Omega} \gamma \nabla u \cdot \nabla \varphi \quad (1.3)$$

where $\varphi \in W^{1,2}(\Omega)$ is a function such that $\varphi|_{\partial\Omega} = \varphi_0$ in the sense of traces.

Since the foundational work of Calderón, the research of the question has been very intense but it is not until 2006 when, by means of quasiconformal mappings, K. Astala and L. Päiväranta in [11] were able to establish the injectivity of the map

$$\gamma \rightarrow \Lambda_\gamma$$

for an arbitrary L^∞ function bounded away from zero. Previous planar results were obtained in [39] and [48]. In higher dimensions, the known results on uniqueness require some extra a priori regularity on γ (basically some control on $\frac{3}{2}$ derivatives of γ , see [47], [18], [41] and [21].)

A relevant question (specially in applications) is the stability of the inverse problem, that is, the continuity of the inverse map

$$\Lambda_\gamma \rightarrow \gamma.$$

For dimension $n > 2$, the known results are due to Alessandrini [4], [5]. There the author proved stability under the extra assumption $\gamma \in W^{2,\infty}$. In the planar case, $n = 2$, the situation is different. Liu proved stability for conductivities in $W^{2,p}$ with $p > 1$ in [36]. In [14], stability was obtained when $\gamma \in C^{1+\alpha}$ with $\alpha > 0$. Recently, Barceló, Faraco and Ruiz [15] obtained stability under the weaker assumption $\gamma \in C^\alpha$, $0 < \alpha < 1$. Precisely, they prove that for any two conductivities γ_1, γ_2 on a Lipschitz domain Ω , with a priori bounds $\frac{1}{K} \leq \gamma_i \leq K, K \geq 1$ and $\|\gamma_i\|_{C^\alpha} \leq \Lambda_0$, the following estimate holds:

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq V(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)})$$

with $V(t) = C \log(\frac{1}{t})^{-a}$. Here $C, a > 0$ depend only on K, α and Λ_0 .

An example, due to Alessandrini [4], shows that in absence of continuity, L^∞ estimates do not hold. Namely, if we denote by $B_{r_0} = \{x \in \mathbf{R}^2, |x| < r_0\}$ the ball centered at the origin with radius r_0 , take $\Omega = B_1$ the unit ball in \mathbf{R}^2 , $\gamma_1 = 1$ and $\gamma_2 = 1 + \chi_{B_{r_0}}$, then $\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} = 1$, but $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{\frac{1}{2}} \rightarrow H^{-\frac{1}{2}}} \leq 2r_0 \rightarrow 0$ as $r_0 \rightarrow 0$.

A closer look to the previous example shows that $\lim_{r_0 \rightarrow 0} \|\gamma_1 - \gamma_2\|_{L^2(\Omega)} = 0$. Therefore one could conjecture that, in absence of continuity, average stability (in the L^2 sense) might hold. However, it is well known that some control on the oscillation of γ is needed to obtain stability. Namely, let γ be defined in the unit square and extended periodically. Then the sequence $\{\gamma(jx)\}_{j=1}^\infty$ G -converges to a matrix γ_0 (see for example [50] for the notion of G -convergence). On one hand, $\gamma(jx)$ has not any convergent subsequence in L^2 . On the other hand, G -convergence implies the convergence of the fluxes [50, Proposition 9]. That is, if u_j, u_0 solve the corresponding Dirichlet

problems for a fixed function $f \in H^{\frac{1}{2}}(\partial\Omega)$,

$$\begin{cases} \nabla \cdot (\gamma_j \nabla u_j) = 0 \\ u_j|_{\partial\Omega} = f \end{cases} \quad (1.4)$$

then, the fluxes satisfy that $\gamma_j u_j \rightharpoonup \gamma \nabla u$. Thus, by (1.3) $\lim_{j_1, j_2 \rightarrow \infty} \langle \Lambda_{\gamma_{j_1}} - \Lambda_{\gamma_{j_2}}, \varphi_0 \rangle$ for each φ_0 . Notice that γ_j can be chosen even being C^∞ , so the problem here is not so much a matter of regularity but rather a control on the oscillation.

In this paper we prove that L^2 stability holds if we prescribe a bound of γ in any fractional Sobolev space $W^{\alpha,2}$. By the relation with Besov spaces this could be interpreted as controlling the average oscillation of the function. Thus average control on the oscillation of the coefficients yields average stability of the inverse problem.

Theorem 1.1. *Let Ω be a Lipschitz domain in the plane. Let $\gamma = \gamma_1, \gamma_2$ be two planar conductivities in Ω satisfying*

- (I) *Ellipticity:* $\frac{1}{K} \leq \gamma(x) \leq K$.
- (II) *Sobolev regularity:* $\gamma_i \in W^{\alpha,p}(\Omega)$ with $\alpha > 0, 1 < p < \infty$, and $\|\gamma_i\|_{W^{\alpha,p}(\Omega)} \leq \Gamma_0$.

Let $\tilde{\alpha} = \min\{\alpha, \frac{1}{2}\}$. Then there exists two constants $c(K, p)$, $C(K, \alpha, p, \Gamma_0) > 0$, such that:

$$\|\gamma_1 - \gamma_2\|_{L^2(\Omega)} \leq \frac{C}{|\log(\rho)|^{c\tilde{\alpha}^2}} \quad (1.5)$$

where $\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)}$.

The theorem is specially interesting for $\alpha \rightarrow 0$. Then we are close to get stability for conductivities in L^∞ . An estimate for the behavior of C in terms of Γ_0 is obtained in the particular case $p = 2$ (see Corollary 6.3), and analogous results for $p \neq 2$ can be deduced by interpolation. Also interpolating one can obtain L^p stability estimates, whose behavior in α will be quadratic as well.

Concerning the logarithmic modulus of continuity, the arguments of Mandache [38] can be adapted to the L^2 setting. Namely we can consider the same set of conductivities with the obvious replacement of the C^m function by a normalized $W^{\alpha,2}$ function. The argument shows the existence of two conductivities such that $\|\gamma_1 - \gamma_2\|_{L^\infty(\mathbb{D})} \leq \epsilon$, $\|\gamma_i\|_{W^{\alpha,p}(\Omega)} \leq \Gamma_0$, but

$$\|\gamma_1 - \gamma_2\|_{L^2(\mathbb{D})} \geq \frac{1}{C |\log(\rho)|^{\frac{3(1+\alpha)}{2\alpha}}}. \quad (1.6)$$

Here C is a constant depending on all the parameters. Notice that the power is better than in the L^∞ setting but still the modulus of continuity is far from being satisfactory.

In our way to prove Theorem 1.1 we have dealt with several questions related to quasiconformal mappings of independent interest. More precisely, we have needed to understand how quasiconformal mappings interact with fractional Sobolev spaces. In particular we analyze the regularity of Beltrami equations with Sobolev bounds on the coefficients which has been a recent topic of interest in the theory. See [24, 25] where the case $\mu \in W^{1,p}$ is investigated in relation with the size of removable sets. We prove the following regularity result.

Theorem 1.2. *Let $\alpha \in (0, 1)$, and suppose that $\mu, \nu \in W^{\alpha,2}(\mathbb{C})$ are Beltrami coefficients, compactly supported in \mathbb{D} , such that*

$$||\mu(z)| + |\nu(z)|| \leq \frac{K-1}{K+1}.$$

at almost every $z \in \mathbb{D}$. Let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be the only homeomorphism satisfying

$$\bar{\partial}\phi = \mu \partial\phi + \nu \bar{\partial}\bar{\phi}$$

and $\phi(z) - z = \mathcal{O}(1/z)$ as $|z| \rightarrow \infty$. Then, $\phi(z) - z$ belongs to $W^{1+\theta\alpha,2}(\mathbb{C})$ for every $\theta \in (0, \frac{1}{K})$, and

$$\|D^{1+\theta\alpha}(\phi - z)\|_{L^2(\mathbb{C})} \leq C_K \left(\|\mu\|_{W^{\alpha,2}(\mathbb{C})}^\theta + \|\nu\|_{W^{\alpha,2}(\mathbb{C})}^\theta \right)$$

for some constant C_K depending only on K .

Many corolaries can be obtained form this theorem by interpolation, as for example what do you obtain if μ is a function of bounded variation. We have contented ourselves with the L^2 setting but similar results hold in L^p . As a consequence of this theorem, we obtain the corresponding regularity of the complex geometric optics solutions.

The other crucial ingredient in our proof is the regularity of $\mu \circ \psi$ where ψ is a normalized quasiconformal mapping. It is well known that quasiconformal mappings preserve BMO and $\dot{W}^{1,2}$ but it is not clear what happens with the intermediate spaces. We prove the following stament,

$$\mu \in W^{\alpha,2} \quad \Rightarrow \quad \mu \circ \psi \in W^{\beta,2}, \quad \text{for every } \beta < \frac{\alpha}{K} \quad (1.7)$$

which suffices for our purposes. The proof relies on the fact that Jacobians of quasiconformal mappings are Muckenhoupt weights [10]

The Lipschitz regularity of the domain Ω is used to reduce the problem to the unit disk \mathbb{D} . This reduction relies on two facts. First, any

Lipschitz domain Ω is an extension domain for fractional Sobolev spaces. Secondly, the characteristic function χ_Ω belongs to $W^{\alpha,2}(\mathbb{C})$ for any $\alpha < \frac{1}{2}$. Indeed, this is responsible also of the constraint $\tilde{\alpha} < \frac{1}{2}$ at Theorem 1.1. In fact, a stability result holds as well if Ω is any simply connected extension domain. To see this, recall that planar simply connected extension domains Ω are quasidisks ([28]), that is, $\Omega = \phi(\mathbb{D})$ where $\phi : \mathbb{C} \rightarrow \mathbb{C}$ is quasiconformal. Therefore, for instance by our results in Section 4, $\chi_\Omega = \chi_\mathbb{D} \circ \phi^{-1}$ belongs to some space $W^{\tilde{\alpha},2}$, and then use Theorem 1.1.

The rest of the paper is organized as follows. In Section 2 we recall previous facts from [11, 15] which will be needed in the present paper, and describe the strategy of our proof. In Section 3 we reduce the problem to conductivities γ such that $\gamma - 1 \in W_0^{\alpha,2}(\mathbb{D})$. In Section 4 we study the interaction between quasiconformal mappings and fractional Sobolev spaces. Finally in Section 5 we prove the subexponential growth of the complex geometric optic solutions and in Section 6 we prove the theorem.

In closing we remark several issues raised by our work. The first one is to improve the logarithmic character of the stability. It was proved by Alessandrini and Vesella that often a logarithmic estimate yields Lipschitz stability for some finite dimensional spaces of conductivities. However, to achieve the desired estimates in our setting seems to require a more subtle understanding of the Beltrami equation and we leave it for the future. It will also be desirable to obtain L^p estimates in terms of $W^{\alpha,p}$ with constants independent of p , so that the C^α situation in [15] could be understood as a limit of this paper. This seems to require an L^2 version of the boundary recovery results of Alessandrini [5] and Brown (see [19]). Finally, from the quasiconformal point of view, there seems to be room for improvement in our estimates specially concerning the composition which is far from being optimal when $\alpha \nearrow 1$, since $\dot{W}^{1,2}$ is invariant under composition with quasiconformal maps. This will also be the issue for further investigations.

Notation

Complex and real derivatives are denoted by

$$\begin{aligned}\partial_{\bar{z}} &= \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \\ \partial_z &= \partial = \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)\end{aligned}$$

where $z = x + iy$. For a mapping $\phi : \Omega \rightarrow \mathbb{C}$, its Jacobian determinant is denoted by $J(z, \phi) = |\partial_z \phi(z)|^2 - |\partial_{\bar{z}} \phi(z)|^2$. For $k \in \mathbb{C}$ we will use the unimodular function $e_k(z) = e^{ikz + i\bar{k}\bar{z}}$. Notice that then we can define the

Fourier transform by

$$\widehat{f}(k) = \int_{\mathbb{C}} e_{-k}(z) f(z) dA(z).$$

The spaces $L^p(\Omega)$, $\dot{W}^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ are defined as usually. Then, following Adams [1], one introduces $W^{\alpha,p}(\Omega)$ as the complex interpolation space

$$W^{\alpha,p}(\Omega) = [L^p(\Omega), W^{1,p}(\Omega)]_{\alpha},$$

and similarly for the homogeneous case $\dot{W}^{\alpha,p}(\Omega) = [L^p(\Omega), \dot{W}^{1,p}(\Omega)]_{\alpha}$. The Hölder space $C^{\alpha}(\Omega)$ over a domain Ω is

$$C^{\alpha}(\Omega) = \left\{ f : \|f\|_{L^{\infty}} + \sup_{x,y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} < \infty \right\}.$$

For simplicity, $H^1(\Omega) = W^{1,2}(\Omega)$ and $H_0^1 = W_0^{1,2}(\Omega)$. By $H^{\frac{1}{2}}(\partial\Omega)$ we denote the quotient space $H^1(\Omega)/H_0^1(\Omega)$. Given a Banach space X we denote the operator norm of $T: X \rightarrow X$ by $\|T\|_X$. We remark that C or a denote constants which may change at each occurrence. We will indicate the dependence of the constants on parameters K, Γ , etc, by writing $C = C(K, \Gamma, \dots)$. Finally, for two conductivities γ_1 and γ_2 , we write

$$\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2} \rightarrow H^{-1/2}}.$$

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2 Scheme of the proof

We will follow the strategy of [15]. This work focusses on the approach based on the Beltrami equation initiated in [11]. The starting point is the answer to Calderón conjecture in the plane obtained by Astala and Päiväranta.

Theorem 2.1 (Astala-Päiväranta). *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain, and let $\gamma_i \in L^{\infty}(\Omega)$, $i = 1, 2$. Suppose that there exist a constant $K > 1$ such that $\frac{1}{K} \leq \gamma_i \leq K$. If*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$$

then $\gamma_1 = \gamma_2$.

In other words, the mapping $\gamma \mapsto \Lambda_\gamma$ is injective. We recall the basic elements from [11] needed in the sequel, also the strategies for uniqueness and stability, and what we will need in the current paper.

Equivalence between Beltrami and conductivity equation: Let \mathbb{D} be the unit disc. If a function u is γ -harmonic in \mathbb{D} , then there exists another function v , called its γ -harmonic conjugate (and actually γ^{-1} -harmonic in Ω), unique modulo constants, such that $f = u + iv$ satisfies the \mathbb{R} -linear Beltrami type equation

$$\bar{\partial}f = \mu \bar{\partial}f \quad (2.1)$$

with

$$\mu = \frac{1 - \gamma}{1 + \gamma} \in \mathbb{R}. \quad (2.2)$$

Then if $K \geq 1$ is the ellipticity constant of γ we denote by

$$\kappa = \frac{K - 1}{K + 1}.$$

It is an algebraic fact to show that $\|\mu\|_\infty \leq \kappa$ and thus the Beltrami equation is elliptic when so is the conductivity equation and viceversa. Moreover, for $x \in (\frac{1}{K}, K)$, the function $F(x) = \frac{1-x}{1+x}$ satisfies $\frac{2}{1+K} \leq |F'(x)| \leq \frac{2K}{1+K}$. Thus, it also follows that

$$\frac{1}{C} \|\gamma\|_{W^{\alpha,p}(\Omega)} \leq \|\mu\|_{W^{\alpha,p}(\Omega)} \leq C \|\gamma\|_{W^{\alpha,p}(\Omega)},$$

where the constant C only depends on K (see Lemma 3.1). Therefore, bounds in terms of μ and γ are equivalent.

We can argue as well in the reverse direction. If $f \in W_{loc}^{1,2}(\mathbb{D})$ satisfies (2.1) for real μ with $\|\mu\|_\infty \leq \kappa$, then we can write $f = u + iv$ where u and v satisfy

$$\operatorname{div}(\gamma \nabla u) = 0 \quad \text{and} \quad \operatorname{div}(\gamma^{-1} \nabla v) = 0.$$

Thus, it is equivalent to determine either γ or μ , and throughout the paper we will work with either of them indistinctly.

As for holomorphic functions, u and v are related by the corresponding Hilbert transform

$$\mathcal{H}_\mu: H^{\frac{1}{2}}(\partial\mathbb{D}) \rightarrow H^{\frac{1}{2}}(\partial\mathbb{D})$$

defined as

$$\mathcal{H}_\mu(u|_{\partial\mathbb{D}}) = v|_{\partial\mathbb{D}}$$

for real functions, and \mathbb{R} -linearly extended to \mathbb{C} -valued functions by setting $\mathcal{H}_\mu(iu) = i\mathcal{H}_{-\mu}(u)$. Since $\partial_T \mathcal{H}_\mu = \Lambda_\gamma$ it follows [11, Proposition 2.7] that

\mathcal{H}_μ , $\mathcal{H}_{-\mu}$ and $\Lambda_{\gamma^{-1}}$ are uniquely determined by Λ_γ . Accordingly in [15, Proposition 2.2] it is shown that

$$\|\mathcal{H}_{\mu_1} - \mathcal{H}_{\mu_2}\| \lesssim \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|$$

with respect to the corresponding operator norms. In other words, the mapping $\Lambda_\gamma \mapsto \mathcal{H}_\mu$ is Lipschitz continuous independently of the regularity of γ .

Existence of complex geometric optics solutions, scattering transform and ∂_k equations: The theory of quasiconformal mappings and Beltrami operators allow to combine in an efficient way ideas from complex analysis, singular integral operators and degree arguments to prove the existence of *complex geometric optics solutions* with no assumptions on the coefficients.

Theorem 2.2. *Let $\kappa \in (0, 1)$, and let μ be a real Beltrami coefficient, compactly supported in \mathbb{D} , satisfying $\|\mu\|_\infty < \kappa$. For every $k \in \mathbb{C}$ and $p \in (2, 1 + \frac{1}{\kappa})$ the equation*

$$\bar{\partial}f = \mu \bar{\partial}f$$

admits a unique solution $f \in W_{loc}^{1,p}(\mathbb{C})$ of the form

$$f(z) = e^{ikz} M_\mu(z, k)$$

such that $M_\mu(z, k) - 1 = \mathcal{O}(1/z)$ as $|z| \rightarrow \infty$. Moreover,

$$\operatorname{Re} \left(\frac{M_{-\mu}}{M_\mu} \right) > 0$$

and $f_\mu(z, 0) = 1$.

In this context, the proper definition of scattering transform of μ (or of γ) is

$$\tau_\mu(k) = \frac{i}{4\pi} \int_{\mathbb{D}} \frac{\partial}{\partial z} \left(e^{i\bar{k}z} (\overline{f_\mu(z)} - \overline{f_{-\mu}(z)}) \right) dA(z). \quad (2.3)$$

The complex geometric optics solutions $\{u_\gamma, \tilde{u}_\gamma\}$ to the divergence type equation (1.1) are then obtained from the corresponding ones from the Beltrami equation by

$$\begin{aligned} u_\gamma &= \operatorname{Re}(f_\mu) + i \operatorname{Im}(f_{-\mu}) \\ \tilde{u}_\gamma &= \operatorname{Im}(f_\mu) + i \operatorname{Re}(f_{-\mu}), \end{aligned}$$

and they uniquely determine the pair $\{f_\mu, f_{-\mu}\}$ (and viceversa) in a stable way. We consider u_γ as a function of (z, k) . In the z plane, u_γ satisfies the complex γ -harmonic equation,

$$\operatorname{div}(\gamma \nabla u_\gamma) = 0.$$

As a function of k , u_γ is a solution to the following $\bar{\partial}$ -type equation

$$\frac{\partial u_\gamma}{\partial \bar{k}}(z, k) = -i \tau_\mu(k) \overline{u(z, k)}. \quad (2.4)$$

Let us emphasize that $\tau_\mu(k)$ is independent of z .

Strategy for uniqueness: Let γ_1, γ_2 be two conductivities. In [11], the strategy for uniqueness is divided in the following steps:

- (i) Reduction to \mathbb{D} .
- (ii) If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\tau_{\mu_1} = \tau_{\mu_2}$.
- (iii) Step (ii) and (2.4) imply that $u_{\gamma_1} = u_{\gamma_2}$.
- (iii) Finally, condition $u_{\gamma_1} = u_{\gamma_2}$ is equivalent to $Du_{\gamma_1} = Du_{\gamma_2}$, which holds as well if and only if $\gamma_1 = \gamma_2$

First step is relatively easy since there is no regularity of γ to preserve and thus one can extend by 0 in $\mathbb{D} \setminus \Omega$. Second step is dealt with in [11, Proposition 6.1]. It is shown that $\mathcal{H}_{\mu_1} = \mathcal{H}_{\mu_2}$ implies $f_{\mu_1}(z, k) = f_{\mu_2}(z, k)$ for all $k \in \mathbb{C}$ and $|z| > 1$. As a consequence (ii) follows.

The step (iii) is more complex because uniqueness results and a priori estimates for pseudoanalytic equations in \mathbb{C} like (2.4) only hold if the coefficients or the solutions decay fast enough at ∞ . Unfortunately the required decay properties for τ seem to require roughly one derivative for γ . However in [11] it is shown that in the measurable setting at least we obtain subexponential decay. That is, we can write,

$$u_\gamma(z, k) = e^{ik(z + \epsilon_\mu(z, k))} \quad (2.5)$$

for some function $\epsilon = \epsilon_\mu(z, k)$ satisfying

$$\lim_{k \rightarrow \infty} \|\epsilon_\mu(z, k)\|_{L^\infty(\mathbb{C})} = 0.$$

This would not be enough if we would consider equation (2.4) for a single z . However, in [11] it is used that $u(z, k)$ solves an equation for each z . Further, one has asymptotic estimates for u both in the k (as above) and z variables. Then, a clever topological argument in both variables shows that, with these estimates, τ_μ determines the solution to (2.4).

Strategy for stability: In order to obtain stability, the natural idea is to try to quantify in an uniform way the arguments for uniqueness. This was done in [15] for C^α conductivities. Let us recall the argument and specially the results which did not require regularity of γ and would be instrumental

for the current work. Let $\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|$. First one reduces to the unit disk by an argument which involves Whitney extension operator, the weak formulation (1.3) and a result of Brown about recovering continuous conductivities at the boundary ([19]). Next we investigate the relation between the corresponding scattering transforms.

Theorem 2.3 (Stability of the scattering transforms). *Let γ_1, γ_2 be conductivities in \mathbb{D} , with $\frac{1}{K} \leq \gamma_i \leq K$, and denote $\mu_i = \frac{1-\gamma_i}{1+\gamma_i}$. Then, for every $k \in \mathbb{C}$ it holds that*

$$|\tau_{\mu_1}(k) - \tau_{\mu_2}(k)| \leq c e^{c|k|} \rho. \quad (2.6)$$

where the constant c depends only on K .

The estimate is just pointwise but on the positive side it holds for L^∞ conductivities. In [15, Theorem 4.6] it is also given an explicit formula for the difference of scattering transforms which might be of independent interest. Next we state a result that is implicitly proved in [15, Theorem 5.1]. There it is stated as a property of solutions to regular conductivities. However, in the proof the regularity is only used to obtain the decay in the k variable. Because of this, here we state it separately as condition (2.7).

Theorem 2.4 (A priori estimates in terms of scattering transform). *Let $K \geq 1$ and γ_1, γ_2 be conductivities on \mathbb{D} , with $\frac{1}{K} \leq \gamma_i \leq K$. Let*

$$u_{\gamma_j}(z, k) = e^{ik(z + \epsilon_{\mu_j}(z, k))},$$

denote, as in (2.5), the complex geometric optics solutions to (1.1). Let us assume that there exist positive constants α, B such that for each $z, k \in \mathbb{C}$,

$$|\epsilon_{\mu_i}(z, k)| \leq \frac{B}{|k|^\alpha}. \quad (2.7)$$

Then it follows that:

A *There exists new constants $b = b(K)$, $C = C(K, B)$, such that for every $z \in \mathbb{C}$ there exists $w \in \mathbb{C}$ satisfying:*

- (a) $|z - w| \leq CB \left| \log \frac{1}{\rho} \right|^{-b\alpha}$, where $\rho = \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|$.
- (b) $u_{\gamma_1}(z, k) = u_{\gamma_2}(w, k)$.

B *For each $k \in \mathbb{C}$, there exists new constants $b = b(K)$ and $C = C(k, K)$ such that*

$$\|u_{\gamma_1}(z, k) - u_{\gamma_2}(z, k)\|_{L^\infty(\mathbb{D}, dA(z))} \leq \frac{CB^{\frac{1}{K}}}{|\log(\rho)|^{b\alpha}}. \quad (2.8)$$

Proof. The proof of **A** follows from [15, Proposition 5.2] and [15, Proposition 5.3]. Let us prove **B**. Given $z \in \mathbb{C}$, let $w \in \mathbb{C}$ be given by part **A**. Then

$$|u_{\gamma_1}(z, k) - u_{\gamma_2}(z, k)| = |u_{\gamma_1}(z, k) - u_{\gamma_1}(w, k)|.$$

By the Hölder continuity of K -quasiregular mappings, together with (a), we get

$$|u_{\gamma_1}(z, k) - u_{\gamma_2}(z, k)| \leq C(k, K) |z - w|^{\frac{1}{K}} \leq C(k, K) C^{\frac{1}{K}} B^{\frac{1}{K}} \left| \log \frac{1}{\rho} \right|^{-\frac{b\alpha}{K}}$$

and the desired estimate follows after renaming the constants. \square

Unlike in the uniqueness arguments, going from $u_{\gamma_1} - u_{\gamma_2}$ to $D(u_{\gamma_1} - u_{\gamma_2})$ is more delicate in the stability setting, since functions do not control their derivatives in general. This is solved in [15], under Hölder regularity, using the following fact.

Theorem 2.5 (Schauder estimates). *Let γ_i , $i = 1, 2$ be conductivities on \mathbb{D} , such that $\frac{1}{K} \leq \gamma_i \leq K$ and $\|\gamma_1\|_{C^\alpha(\mathbb{D})} \leq \Gamma_0$. As always, denote $\mu_i = \frac{1-\gamma_i}{1+\gamma_i}$, and let $f_{\mu_i}(z, k)$ be the corresponding complex geometric optics solutions to (2.1). Then*

1. *For each $k \in \mathbb{C}$ there exists a constant $C = C(k) > 0$ with*

$$\|f_{\mu_1}(\cdot, k) - f_{\mu_2}(\cdot, k)\|_{C^{1+\alpha}(\mathbb{D})} \leq C(k). \quad (2.9)$$

2. *The jacobian determinant of $f_{\mu_i}(z, k)$ has a positive lower bound*

$$J(z, f_{\mu_i}(\cdot, k)) \geq C(K, k, \Gamma_0).$$

Now, to finish the proof of stability for Hölder continuous conductivities, just note that an interpolation argument between L^∞ and $C^{1+\alpha}$ gives Lipschitz bounds for Df_{μ_i} . Thus, by $\mu = \frac{\bar{\partial}f}{\partial f}$ and the second statement above, one obtains L^∞ stability for $\mu_1 - \mu_2$. The corresponding result for $\gamma_1 - \gamma_2$ comes due to (2.2).

Strategy for stability under Sobolev regularity In the current work we will try to push the previous strategy to obtain L^2 stability. The previous analysis shows that we can rely in many of the results from [11, 15]. In particular, we only have to prove that $\tau_\mu \mapsto \mu$ is continuous.

For this, we start by reducing the problem in Section 3. We replace the assumption $\gamma_i \in W^{\alpha,p}(\Omega)$ by $\gamma_i \in W_0^{\beta,2}(\mathbb{D})$, where $0 < \beta < \min\{\frac{1}{2}, \alpha\}$. For this, it is used there that characteristic functions of Lipschitz domains belong to $W^{\beta,q}(\mathbb{C})$ whenever $\beta q < 1$.

Then we follow by investigating the regularity of solutions of Beltrami equations with coefficients in fractional Sobolev spaces in order to obtain an estimate like (2.9), with the $\mathcal{C}^{1+\alpha}$ norm replaced by the sharp Sobolev norm attainable under our assumption on the Beltrami coefficient (see Theorem 4.6). It is also needed here to understand how composition with quasiconformal mappings affects fractional Sobolev spaces. As far as we know, the estimates here are new and of their own interest.

Afterwards we prove that our Sobolev assumption on μ suffices to get the uniform subexponential growth of the geometric optics solutions needed in condition (2.7) in Theorem 2.4 (this is done in Section 5, see Theorem 5.7). In fact we obtain a very clean expression for the precise growth, achieving that the exponent depends linearly on α . Finally, in Section 6 we do the interpolation argument. Here we do not have enough regularity to control $W^{1,\infty}$ norms and here is where one sees why we need to be happy with the control on $\|\mu_1 - \mu_2\|_{L^2(\mathbb{D})}$. Also we do not have a pointwise lower bound for the corresponding Jacobians which causes also difficulties.

3 Fractional Sobolev spaces and Reduction to $\mu \in W_0^{\alpha,2}(\mathbb{D})$

3.1 On fractional Sobolev Spaces

Following [1, p.21], for any domain Ω , we denote by $W^{1,p}(\Omega)$ the class of $L^p(\Omega)$ functions f with $L^p(\Omega)$ distributional derivatives of first order. This means that for any constant coefficients first order differential operator D there exists an $L^2(\Omega)$ function Df such that

$$\int_{\Omega} f D\varphi = - \int_{\Omega} Df \varphi$$

whenever $\varphi \in \mathcal{C}^\infty$ is compactly supported inside of Ω . Similarly one can define the Sobolev spaces $W^{m,p}(\Omega)$ of general integer order $m \geq 1$.

It comes from the work of Calderón (see [2, p.7] or [45]) that every Lipschitz domain Ω is an *extension domain*. That is, for any integer $m > 0$ and any domain $\Omega' \supset \overline{\Omega}$ there exists a bounded linear extension operator

$$E_m : W^{m,p}(\Omega) \rightarrow W_0^{m,p}(\Omega')$$

and therefore for every function $f \in W^{m,2}(\Omega)$ there is another function $E_m f \in W^{m,p}(\Omega')$ such that $E_m f|_{\Omega} = f$. Of course, $E_m f \in W^{1,p}(\mathbb{C})$.

Let us introduce for general domains Ω and any real number $0 < \alpha < 1$ the complex interpolation space

$$W^{\alpha,p}(\Omega) = [L^p(\Omega), W^{1,p}(\Omega)]_{\alpha}.$$

The closure of $\mathcal{C}_0^\infty(\Omega)$ (\mathcal{C}^∞ functions with compact support contained in Ω) in $W^{\alpha,p}(\Omega)$ is denoted by $W_0^{\alpha,p}(\Omega)$. Functions in $W_0^{\alpha,p}(\Omega)$ can be extended by zero to the whole plane, and the extension belongs to $W^{\alpha,p}(\mathbb{C})$. Thus, we can identify any function in $W_0^{\alpha,p}(\Omega)$ with its extension in $W^{\alpha,p}(\mathbb{C})$.

When Ω is an extension domain, an interpolation argument shows (see [1, p.222]) that $W^{\alpha,p}(\Omega)$ coincides with the space of restrictions to Ω of functions in $W^{\alpha,p}(\mathbb{C})$. That is, to each function $u \in W^{\alpha,p}(\Omega)$ one can associate a function $\tilde{u} \in W^{\alpha,p}(\mathbb{C})$ such that $\tilde{u}|_\Omega = u$ and $\|\tilde{u}\|_{W^{\alpha,p}(\mathbb{C})} \leq C \|u\|_{W^{\alpha,p}(\Omega)}$.

We have chosen just one way to introduce the fractional Sobolev spaces. In the rest of the subsection, we discuss the alternative characterizations and properties of these spaces needed in the rest of the paper. Two good sources for the basics of this theory are [1, Chapter 7], [45, Chapter 4].

Fourier side. For $p = 2$, it is easy to see that,

$$W^{\alpha,2}(\mathbb{C}) = \left\{ f \in L^2(\mathbb{C}); (1 + |\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi) \in L^2(\mathbb{C}) \right\}$$

and that this agrees with the space of Bessel potentials

$$W^{\alpha,2}(\mathbb{C}) = \{ f = G_\alpha * g; g \in L^2(\mathbb{C}) \}$$

where G_α is the Bessel Kernel [2, p.10]. For $p \neq 2$ the situation is more complicated but it can be shown that

$$W^{\alpha,p}(\mathbb{C}) = \left\{ f \in L^p(\mathbb{C}); \left((1 + |\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi) \right)^\wedge \in L^p(\mathbb{C}) \right\}.$$

Integral modulus of continuity We define the L^p -difference of a function f by

$$\omega_p(f)(y) = \|f(\cdot + y) - f(\cdot)\|_{L^p(\mathbb{C})}. \quad (3.1)$$

(see [45, Chapter V] Then the Besov spaces $B_\alpha^{p,q}(\mathbb{R}^n)$ are defined by

$$B_\alpha^{p,q}(\mathbb{R}^n) = \{ f \in L^p(\mathbb{R}^n) : \int_{\mathbb{R}^n} \omega_p(f)(y)^q |y|^{-(n+\alpha q)} < \infty \}.$$

There are many relations between Besov and fractional Sobolev spaces. We will need the following two facts,

$$B_\alpha^{2,2} = W^{\alpha,2}, \quad W^{\alpha,p} \subset B_\alpha^{p,2} \quad (p < 2). \quad (3.2)$$

For a proof see [1, Chapter 7] or [45, Chapter V].

Leibniz Rule [[31]]

Lemma 3.1. *Let $\alpha \in (0, 1)$ and $p \in (1, \infty)$.*

(a) *Let $f, g \in \mathcal{C}_0^\infty(\mathbb{C})$. Then,*

$$\|D^\alpha(fg) - f D^\alpha(g) - g D^\alpha(f)\|_p \leq C \|D^{\alpha_1}(f)\|_{p_1} \|D^{\alpha_2}(g)\|_{p_2}$$

whenever $\alpha_1, \alpha_2 \in [0, \alpha]$ are such that $\alpha_1 + \alpha_2 = \alpha$ and $p_1, p_2 \in (1, \infty)$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

(b) *Let $f, g \in \mathcal{C}_0^\infty(\mathbb{C})$. Then*

$$\|D^\alpha(f \circ g)\|_p \leq C \|Df(g)\|_{p_1} \|D^\alpha g\|_{p_2}$$

whenever $p_1, p_2 \in (1, \infty)$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

(c) *Let $f, g \in \mathcal{C}_0^\infty(\mathbb{C})$. Then*

$$\|D^\alpha(fg) - f D^\alpha(g) - g D^\alpha(f)\|_p \leq C \|D^\alpha(f)\|_p \|g\|_\infty$$

whenever $0 < \alpha < 1$ and $1 < p < \infty$.

Remark 3.2. From property (a) and (c) it follows the generalized Leibniz rule

$$\|D^\alpha(fg)\|_p \leq \|D^\alpha f\|_{p_1} \|g\|_{p_2} + \|D^\alpha g\|_{p_3} \|f\|_{p_4} \quad (3.3)$$

whenever $1 \leq p_1, p_2, p_3, p_4 \leq \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. Moreover if $\text{suppt}(f, g) \in \mathbb{D}$ we have that

$$\|D^\alpha(fg)\|_p \leq \|D^\alpha f\|_{p_1} \|g\|_{p_2} + \|D^\alpha g\|_{L^{p_3}(\mathbb{D})} \|f\|_{L^{p_4}(\mathbb{D})} \quad (3.4)$$

Pointwise Inequalities

Lemma 3.3 (Pointwise inequalities, [46]). *If $f \in W^{\alpha, p}(\mathbb{C})$, $\alpha > 0$, $1 < p < \infty$, then for each $0 < \lambda < \alpha$ there exists a function $g = g_\lambda \in L^{p_\lambda}(\mathbb{C})$, $p_\lambda = \frac{2p}{2-(\alpha-\lambda)p}$ such that*

$$|f(z) - f(w)| \leq |z - w|^\lambda (g(z) + g(w)) \quad (3.5)$$

for almost every $z, w \in \mathbb{C}$. Furthermore, we have that

$$\|g\|_{L^{p_\lambda}(\mathbb{C})} \leq C_\lambda \|f\|_{W^{\alpha, p}(\mathbb{C})},$$

and the constant C_λ remains bounded as $\lambda \rightarrow \alpha$.

3.2 Reduction to $p = 2$

This reduction relies on the fact that $\mu \in L^\infty(\mathbb{C}) \cap W^{\alpha,p}(\mathbb{C})$ and the following interpolation Lemma.

Lemma 3.4. *Let $f \in W^{\alpha_0,p_0} \cap W^{\alpha_1,p_1}$, where $1 < p_0, p_1 < \infty$, $0 \leq \alpha_0, \alpha_1 \leq 1$, and $\theta \in (0, 1)$. Then,*

$$\|f\|_{W^{\alpha,p}} \leq \|f\|_{W^{\alpha_0,p_0}}^\theta \|f\|_{W^{\alpha_1,p_1}}^{1-\theta}$$

where

$$\alpha = \theta \alpha_0 + (1 - \theta) \alpha_1 \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{p_0} + \frac{1 - \theta}{p_1}.$$

Furthermore, if either $p_0 = \infty$ or $p_1 = \infty$, then the above inequality holds true by replacing W^{α_i,p_i} by the Riesz potentials space $I_{\alpha_i} * BMO$.

Proof. It is well known that the complex interpolation method gives

$$[W^{\alpha_0,p_0}, W^{\alpha_1,p_1}]_\theta = W^{\alpha,p}$$

whenever $1 < p < \infty$ (for the proof of this, see for instance [51]). For $p = \infty$, the same result holds true if we replace $W^{\alpha,\infty}$ by the space of Riesz potentials $I_\alpha * BMO$ of BMO functions (for this, see [43]). \square

Let μ be a compactly supported Beltrami coefficient. Then, it belongs both to $L^1(\mathbb{C})$ and $L^\infty(\mathbb{C})$. If we also assume that $\mu \in W^{\alpha,p}(\mathbb{C})$ for some α, p , then we can use the above interpolation to see that $\mu \in W^{\beta,q}(\mathbb{C})$, for any $1 < q < \infty$ and some $0 < \beta < \alpha$. We are particularly interested in $q = 2$.

Lemma 3.5. *Suppose that $\mu \in W^{\alpha,p}(\Omega) \cap L^\infty(\Omega)$ for some $p > 1$ and $0 < \alpha < 1$. Then,*

- For any $0 \leq \theta \leq 1$,

$$\|\mu\|_{W^{\alpha\theta, \frac{p}{\theta}}(\Omega)} \lesssim \|\mu\|_{L^\infty(\Omega)}^{1-\theta} \|\mu\|_{W^{\alpha,p}(\Omega)}^\theta.$$

- For any $0 \leq \theta \leq 1$,

$$\|\mu\|_{W^{\theta\alpha, \frac{p}{(1-\theta)p+\theta}}(\Omega)} \lesssim \|\mu\|_{L^1(\Omega)}^{1-\theta} \|\mu\|_{W^{\alpha,p}(\Omega)}^\theta.$$

- One always has

$$\|\mu\|_{W^{\beta,2}(\Omega)} \leq C(K, p) \|\mu\|_{W^{\alpha,p}(\Omega)}^{p^*/2},$$

$$\text{where } \beta = \frac{\alpha p^*}{2} \text{ and } p^* = \min\{p, \frac{p}{p-1}\}.$$

Proof. The first inequality comes easily interpolating between $BMO(\Omega)$ and $W^{\alpha,p}(\Omega)$ (see [43] for more details). For the second, simply notice that compactly supported Beltrami coefficients belong to all $L^p(\Omega)$ spaces, $p > 1$, so one can do the same between $L^{1+\varepsilon}(\Omega)$ (ε as small as desired) and $W^{\alpha,p}(\Omega)$. The last statement is obtained by letting $\theta = \frac{p^*}{2}$ above. \square

3.3 Reduction to $\Omega = \mathbb{D}$ and $\mu \in W_0^{\alpha,p}(\mathbb{D})$

The proof of the following lemma relies in the fact that characteristic functions of Lipschitz belong to $W^{\alpha,2}$ for each $\alpha < \frac{1}{2}$.

Theorem 3.6. *Let Ω be a Lipschitz domain, strictly included in \mathbb{D} . Let $\mu \in W^{\alpha,2}(\Omega)$. Define*

$$\tilde{\mu} = \begin{cases} \mu & \Omega \\ 0 & \mathbb{C} \setminus \Omega \end{cases}.$$

Then, $\tilde{\mu} \in W_0^{\beta,2}(\mathbb{C})$ for $\beta < \min\{\alpha, \frac{1}{2}\}$ and

$$\|\tilde{\mu}\|_{W^{\beta,2}(\mathbb{C})} \leq C \|\mu\|_{W^{\alpha,2}(\mathbb{C})}.$$

Analogous results can be stated for the extensions by 1 of γ_i .

Proof. Since Ω is an extension domain, there is an extension μ_0 of μ belonging to $W^{\alpha,2}(\mathbb{C})$. Of course, such extension μ_0 need not be supported in Ω any more. Now $\tilde{\mu}$ can be introduced as the pointwise multiplication

$$\tilde{\mu} = \chi_{\Omega} \mu_0.$$

By virtue Lemma 3.1 it is enough to study the smoothness of the characteristic function χ_{Ω} . A way to see this is to recall that fractional Sobolev spaces are invariant under composition with bilipschitz maps [54]. Now, the characteristic function of the half plane belongs to $W_{loc}^{\alpha,p}(\mathbb{C})$ whenever $\alpha p < 1$. Therefore, by a partition of unity argument, we get that $\chi_{\Omega} \in W^{\alpha,p}(\mathbb{C})$ when $\alpha p < 1$. The proof is conclude. \square

Now we need to compare the original Dirichlet-to-Neumann maps with the Dirichlet-to-Neumann maps of the extensions.

Lemma 3.7. *Let Ω be a domain strictly included in \mathbb{D} . Let $\gamma_1, \gamma_2 \in L^{\infty}(\Omega)$ be conductivities in Ω . Further, assume that*

$$\frac{1}{K} \leq \gamma_i(z) \leq K$$

for almost every $z \in \Omega$. Let $\tilde{\gamma}_i$ denote the corresponding extensions by 1 to all of \mathbb{C} . Then,

$$\|\Lambda_{\tilde{\gamma}_1} - \Lambda_{\tilde{\gamma}_2}\|_{H^{\frac{1}{2}}(\partial\mathbb{D}) \rightarrow H^{-\frac{1}{2}}(\partial\mathbb{D})} \leq C \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)}.$$

Proof. We follow the ideas of [15, Theorem 6.2], although the stability result from [19] is not needed in our situation. Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\mathbb{D})$. Let $\tilde{u}_j \in H^1(\mathbb{D})$ be the solution to

$$\begin{cases} \nabla \cdot (\tilde{\gamma}_j \nabla \tilde{u}_j) = 0 & \text{in } \mathbb{D} \\ \tilde{u}_j = \varphi_0 & \text{in } \partial\mathbb{D}. \end{cases}$$

Let also u_2 be defined by

$$\begin{cases} \nabla \cdot (\gamma_2 \nabla u_2) = 0 & \text{in } \Omega \\ u_2 = \tilde{u}_1 & \text{in } \partial\Omega. \end{cases}$$

Define now $\tilde{v}_2 = u_2 \chi_\Omega + \tilde{u}_1 \chi_{\mathbb{D} \setminus \Omega}$. As in [15], we first control $\tilde{u}_2 - \tilde{v}_2$ in terms of ρ . To do this,

$$\begin{aligned} \int_{\mathbb{D}} |\nabla(\tilde{v}_2 - \tilde{u}_2)|^2 &\leq c \int_{\mathbb{D}} \tilde{\gamma}_2 \nabla(\tilde{v}_2 - \tilde{u}_2) \cdot \nabla(\tilde{v}_2 - \tilde{u}_2) \\ &= c \int_{\mathbb{D}} \tilde{\gamma}_2 \nabla \tilde{v}_2 \cdot \nabla(\tilde{v}_2 - \tilde{u}_2) \end{aligned}$$

because $\tilde{v}_2 - \tilde{u}_2 \in H_0^1(\mathbb{D})$ and the $\tilde{\gamma}_2$ -harmonicity of \tilde{u}_2 in \mathbb{D} . By adding and subtracting $\int_{\mathbb{D}} \tilde{\gamma}_1 \nabla \tilde{u}_1 \cdot \nabla(\tilde{v}_2 - \tilde{u}_2)$, and using that $\tilde{\gamma}_1 = \tilde{\gamma}_2 = 1$ off Ω , the right hand side above is bounded by a constant times

$$\left| \int_{\mathbb{D}} \tilde{\gamma}_1 \nabla \tilde{u}_1 \cdot \nabla(\tilde{v}_2 - \tilde{u}_2) \right| + \left| \int_{\Omega} (\gamma_1 \nabla \tilde{u}_1 - \gamma_2 \nabla u_2) \cdot \nabla(\tilde{v}_2 - \tilde{u}_2) \right|.$$

Here the first term vanishes because \tilde{u}_1 is $\tilde{\gamma}_1$ -harmonic on \mathbb{D} and $\tilde{v}_2 - \tilde{u}_2 \in H_0^1(\mathbb{D})$. For the second, we observe that \tilde{u}_1 is γ_1 -harmonic in Ω , u_2 is γ_2 -harmonic in Ω , and $u_2 - \tilde{u}_1 \in H_0^1(\Omega)$. Thus,

$$\begin{aligned} \left| \int_{\Omega} (\gamma_1 \nabla \tilde{u}_1 - \gamma_2 \nabla u_2) \cdot \nabla(\tilde{v}_2 - \tilde{u}_2) \right| &= \left| \langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2})(\tilde{u}_1|_{\partial\Omega}), (\tilde{v}_2 - \tilde{u}_2)|_{\partial\Omega} \rangle \right| \\ &\leq \rho \|\tilde{u}_1\|_{H^{\frac{1}{2}}(\partial\Omega)} \|\tilde{v}_2 - \tilde{u}_2\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq \rho \|\nabla \tilde{u}_1\|_{L^2(\Omega)} \|\nabla(\tilde{v}_2 - \tilde{u}_2)\|_{L^2(\Omega)} \end{aligned}$$

Summarizing, we get

$$\begin{aligned} \left(\int_{\mathbb{D}} |\nabla(\tilde{v}_2 - \tilde{u}_2)|^2 \right)^{\frac{1}{2}} &\leq c \rho \|\nabla \tilde{u}_1\|_{L^2(\Omega)} \leq c \rho \|\nabla \tilde{u}_1\|_{L^2(\mathbb{D})} \\ &\leq c \rho \|\varphi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}. \end{aligned} \quad (3.6)$$

We will use this to compare the Dirichlet-to-Neumann maps at $\partial\mathbb{D}$. If $\psi_0 \in H^{\frac{1}{2}}(\partial\mathbb{D})$ is any testing function, and ψ is any $H^1(\mathbb{D})$ extension,

$$\langle (\Lambda_{\tilde{\gamma}_1} - \Lambda_{\tilde{\gamma}_2})(\varphi_0), \psi_0 \rangle = \int_{\mathbb{D}} (\tilde{\gamma}_1 \nabla \tilde{u}_1 - \tilde{\gamma}_2 \nabla \tilde{u}_2) \cdot \nabla \psi. \quad (3.7)$$

We will divide the bound of this quantity in two steps. For the first,

$$\left| \int_{\mathbb{D}} (\tilde{\gamma}_1 \nabla \tilde{u}_1 - (\gamma_2 \chi_\Omega + \tilde{\gamma}_1 \chi_{\mathbb{D} \setminus \Omega}) \nabla \tilde{v}_2) \cdot \nabla \psi \right| = \left| \langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2})(\tilde{u}_1|_{\partial\Omega}), \psi|_{\partial\Omega} \rangle \right|$$

which is bounded by

$$\begin{aligned}
\rho \|\tilde{u}_1|_{\partial\Omega}\|_{H^{\frac{1}{2}}(\partial\Omega)} \|\psi|_{\partial\Omega}\|_{H^{\frac{1}{2}}(\partial\Omega)} &\leq \rho \|\nabla \tilde{u}_1\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega)} \\
&\leq \rho \|\nabla \tilde{u}_1\|_{L^2(\mathbb{D})} \|\nabla \psi\|_{L^2(\mathbb{D})} \\
&\leq \rho \|\varphi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})} \|\psi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}.
\end{aligned}$$

We are left with,

$$\left| \int_{\mathbb{D}} ((\gamma_2 \chi_{\Omega} + \tilde{\gamma}_1 \chi_{\mathbb{D}\setminus\Omega}) \nabla \tilde{v}_2 - \tilde{\gamma}_2 \nabla \tilde{u}_2) \cdot \nabla \psi \right|$$

which is smaller than,

$$\left| \int_{\Omega} \gamma_2 \nabla(\tilde{v}_2 - \tilde{u}_2) \cdot \nabla \psi + \int_{\mathbb{D}\setminus\Omega} \nabla(\tilde{v}_2 - \tilde{u}_2) \cdot \nabla \psi \right|$$

which in turn is controlled, using (3.6), by a multiple of

$$\begin{aligned}
\int_{\mathbb{D}} |\nabla(\tilde{v}_2 - \tilde{u}_2)| |\nabla \psi| &\leq \|\nabla(\tilde{v}_2 - \tilde{u}_2)\|_{L^2(\mathbb{D})} \|\nabla \psi\|_{L^2(\mathbb{D})} \\
&\leq c \rho \|\varphi\|_{H^{\frac{1}{2}}(\partial\mathbb{D})} \|\psi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}.
\end{aligned}$$

This gives for (3.7) that the difference of Dirichlet-to-Neumann maps satisfies

$$|\langle (\Lambda_{\tilde{\gamma}_1} - \Lambda_{\tilde{\gamma}_2})(\varphi_0), \psi_0 \rangle| \leq c \rho \|\varphi\|_{H^{\frac{1}{2}}(\partial\mathbb{D})} \|\psi_0\|_{H^{\frac{1}{2}}(\partial\mathbb{D})}$$

as desired. \square

Remark 3.8. The trivial extension of the conductivities by 1 simplifies the arguments but has the prizes or loosing regularity if $\alpha \geq 1/2$. An argument similar to that in [15] would need an L^2 version of the boundary recovery result of Brown (see also [5]) of the type

$$\|\gamma_1 - \gamma_2\|_{L^2(\partial\Omega)} \leq C\rho$$

4 Beltrami equations and fractional Sobolev spaces

This section is devoted to investigate how quasiconformal mappings interplay with fractional Sobolev spaces. We face three different goals. First, given a Beltrami coefficient $\mu \in W_0^{\alpha,2}(\mathbb{C})$, we find $\beta \in (0, \alpha)$ such that for any K -quasiconformal mapping ϕ the composition $\mu \circ \phi$, which is another Beltrami coefficient with the same ellipticity bound, belongs to $W^{\beta,2}(\mathbb{C})$.

Secondly, we obtain the optimal (at least when $\alpha \approx 1$), Sobolev regularity for the homeomorphic solutions to the equation

$$\bar{\partial}f = \mu \partial f + \nu \bar{\partial}f$$

under the assumptions of ellipticity and Sobolev regularity for the coefficients. Finally, we obtain bounds for the complex geometric optics solutions. Many properties of planar quasiconformal mappings rely on two precise integral operators, the Cauchy transform,

$$\mathcal{C}\varphi(z) = \frac{-1}{\pi} \int \frac{\varphi(w)}{(w-z)} dA(w). \quad (4.1)$$

and the Beurling transform,

$$T\varphi(z) = \frac{-1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|w-z| \geq \varepsilon} \frac{\varphi(w)}{(w-z)^2} dA(w). \quad (4.2)$$

Their basic properties are well known and can be found in any reference concerning planar quasiconformal mappings, [3, 9, 11].

4.1 Composition with quasiconformal mappings

Let μ be a compactly supported Beltrami coefficient, satisfying

$$|\mu| \leq \frac{K-1}{K+1} = \kappa \chi_{\mathbb{D}}.$$

Further, assume that

$$\mu \in W^{\alpha,2}(\mathbb{C}) \text{ and } \|\mu\|_{W^{\alpha,2}(\mathbb{C})} \leq \Gamma_0$$

for some $\alpha > 0$ and some $\Gamma_0 > 0$. Let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be a planar K -quasiconformal mapping. In this section, we look for those $\beta > 0$ such that $\mu \circ \phi \in W^{\beta,2}(\mathbb{C})$.

We need to recall a local version of a lemma due to Fefferman and Stein, see [40] and [26, Proposition 2.24]. The proof follows from Vitali covering Lemma, exactly as in [40]. By Mf we denote the Hardy-Littlewood maximal function,

$$Mf(x) = \sup \frac{1}{|D|} \int_D f.$$

where the supremum runs over all disks D with $x \in D$, while $M_{\Omega}f$ denote its local version, that is,

$$M_{\Omega}f(x) = \sup \frac{1}{|D|} \int_D f$$

where the supremum is taken over all discs D with $x \in D \subset \Omega$.

Lemma 4.1. *Let $w \geq 0$ a locally integrable function. Then*

$$\int_{\Omega} |M_{\Omega} f|^p \omega dx \leq \int_{\Omega} |f|^p M\omega.$$

We can now prove the main result of this section.

Proposition 4.2. *Let $K \geq 1$. Let $\mu \in W^{\alpha,2}(\mathbb{C})$ for some $\alpha \in (0,1)$, and assume that $|\mu| \leq \frac{K-1}{K+1} \chi_{|\mathbb{D}|}$. Let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be any K -quasiconformal mapping, conformal out of a compact set, and normalized so that $|\phi(z)-z| \rightarrow 0$ as $|z| \rightarrow \infty$. Then*

$$\mu \circ \phi \in W^{\beta,2}(\mathbb{C})$$

whenever $\beta < \frac{\alpha}{K}$. Moreover,

$$\|\mu \circ \phi\|_{W^{\beta,2}(\mathbb{C})} \leq C \|\mu\|_{W^{\alpha,2}(\mathbb{C})}^{\frac{1}{K}},$$

for some constant $C > 0$ depending only on α, β and K .

Proof. It is clear that $\mu \circ \phi$ belongs to $L^2(\mathbb{C})$, so since $W^{\alpha,2}$ agrees with the Besov space $B_{\alpha}^{2,2}$, it suffices to show the convergence of the integral

$$\int_{\mathbb{C}} \int_{\mathbb{C}} \frac{|\mu(\phi(z+w)) - \mu(\phi(z))|^2}{|w|^{2+2\beta}} dA(z) dA(w)$$

for every $\beta < \frac{\alpha}{K}$. First of all, note that by Koebe's $\frac{1}{4}$ Theorem we have the inclusions $\phi(\mathbb{D}) \subset 4\mathbb{D}$ and $\phi^{-1}(4\mathbb{D}) \subset 16\mathbb{D}$. Thus, for large w there is nothing to say since

$$\begin{aligned} & \int_{|w|>1} \int_{\mathbb{C}} \frac{|\mu(\phi(z+w)) - \mu(\phi(z))|^2}{|w|^{2+2\beta}} dA(z) dA(w) \\ & \leq 2 \|\mu\|_{L^2(\mathbb{C})}^2 \int_{|w|>1} \frac{1}{|w|^{2+2\beta}} dA(w) = \frac{C \|\mu\|_{L^2(\mathbb{C})}^2}{\beta} \end{aligned}$$

for some universal constant $C > 0$. Then we are left to bound the integral

$$\int_{|w|\leq 1} \int_{\mathbb{C}} \frac{|\mu(\phi(z+w)) - \mu(\phi(z))|^2}{|w|^{2+2\beta}} dA(z) dA(w).$$

Notice that this integral is in fact the same as

$$\int_{|w|\leq 1} \int_F \frac{|\mu(\phi(z+w)) - \mu(\phi(z))|^2}{|w|^{2+2\beta}} dA(z) dA(w) \quad (4.3)$$

where $F = \{z \in \mathbb{C} : d(z, \phi^{-1}(\mathbb{D})) \leq 1\} \subset 17\mathbb{D}$ by Koebe's Theorem again.

Since $\mu \in W^{\alpha,2}(\mathbb{C})$, then by interpolation we get that $\mu \in W^{\alpha\theta, \frac{2}{\theta}}(\mathbb{C})$ for each $\theta \in (0,1)$ and

$$\|\mu\|_{W^{\alpha\theta, \frac{2}{\theta}}(\mathbb{C})} \leq C \|\mu\|_{\infty}^{1-\theta} \|\mu\|_{W^{\alpha,2}(\mathbb{C})}^{\theta}.$$

The goal is to choose θ to obtain larger possible β . For this, we use condition (3.5). Indeed, for each $\lambda \in (0, \alpha\theta)$ there exists a function $g = g_{\lambda} \in L^{p_{\lambda}}(\mathbb{C})$, $p_{\lambda} = \frac{2}{\theta - (\alpha\theta - \lambda)}$, such that

$$|\mu(\zeta) - \mu(\xi)| \leq |\zeta - \xi|^{\lambda} (g(\zeta) + g(\xi))$$

at almost every $\zeta, \xi \in \mathbb{C}$. Furthermore,

$$\begin{aligned} \|g_{\lambda}\|_{L^{p_{\lambda}}(\mathbb{C})} &\leq C_{\lambda} \|\mu\|_{W^{\alpha\theta, \frac{2}{\theta}}(\mathbb{C})} \\ &\leq C_{\lambda} \|\mu\|_{\infty}^{1-\theta} \|\mu\|_{W^{\alpha,2}(\mathbb{C})}^{\theta}. \end{aligned}$$

with C_{λ} bounded as $\lambda \rightarrow \alpha\theta$. Hence, if $|w| \leq 1$ then

$$\frac{|\mu(\phi(z+w)) - \mu(\phi(z))|}{|w|^{\lambda}} \leq \left(\frac{|\phi(z+w) - \phi(z)|}{|w|} \right)^{\lambda} (g(\phi(z+w)) + g(\phi(z))).$$

Now use quasiconformality and the reverse Hölder inequality for the jacobian (see [10] for a more precise formulation) to get that

$$\begin{aligned} \left(\frac{|\phi(z+w) - \phi(z)|}{|w|} \right)^{\lambda} &\leq C_K \left(\frac{\text{diam } \phi(D(z, |w|))}{\text{diam } D(z, |w|)} \right)^{\lambda} \\ &\leq C_K \left(\frac{1}{|D(z, |w|)|} \int_{D(z, |w|)} J(\zeta, \phi) dA(\zeta) \right)^{\frac{\lambda}{2}} \\ &\leq C_K (M_{\Omega} J_{\lambda}(z))^{\frac{1}{2}} \end{aligned}$$

where $\Omega = \{z \in \mathbb{C} : d(z, \phi^{-1}(\mathbb{D})) \leq 2\}$ and $M_{\Omega} J_{\lambda}(z)$ denotes the local Hardy-Littlewood maximal function M_{Ω} at the point z of $J(\cdot, \phi)^{\lambda}$. Note also that $\Omega \subset 18\mathbb{D}$ by Koebe's Theorem. By symmetry, we could also write $M_{\Omega} J_{\lambda}(z+w)$ instead of $M_{\Omega} J_{\lambda}(z)$, so we end up getting

$$\frac{|\mu(\phi(z+w)) - \mu(\phi(z))|^2}{|w|^{2\lambda}} \leq C (M_{\Omega} J_{\lambda}(z+w) g(\phi(z+w))^2 + M_{\Omega} J_{\lambda}(z) g(\phi(z))^2).$$

Therefore the integral at (4.3) is bounded by a constant times

$$\int_{|w| \leq 1} \int_F \frac{M_{\Omega} J_{\lambda}(z+w) g(\phi(z+w))^2 + M_{\Omega} J_{\lambda}(z) g(\phi(z))^2}{|w|^{2+2\beta-2\lambda}} dA(z) dA(w).$$

If we restrict ourselves to values of λ within the interval $(\beta, \alpha\theta)$, then the integral above is bounded by

$$\frac{C}{\lambda - \beta} \int_F M_{\Omega} J_{\lambda}(z) g(\phi(z))^2 dA(z). \quad (4.4)$$

To get bounds for this, we start by choosing parameters. Fix α, β and K with $\beta < \frac{\alpha}{K}$. Then we can find $s > 1$ such that $\beta s < \frac{\alpha}{K}$. Now let us consider real numbers $\theta \in (0, 1)$ and $\lambda \in (\beta, \alpha\theta)$ satisfying

$$\lambda + (1 - \alpha)\theta < \frac{1}{Ks}. \quad (4.5)$$

Such conditions are compatible precisely when $\beta s < \frac{\alpha}{K}$. Condition (4.5) also guarantees that $p_\lambda > 2Ks$, so we can find r satisfying

$$1 + \lambda s(K - 1) < r < \frac{p_\lambda}{2Ks}(1 + \lambda s(K - 1)), \quad (4.6)$$

and obviously $r > 1$.

Once the parameters have been chosen, we proceed as follows. First, by Hölder's inequality

$$\begin{aligned} \int_F M_\Omega J_\lambda(z) g \circ \phi(z)^2 dA(z) &= \int_\Omega M_\Omega J_\lambda(z) g \circ \phi(z)^2 \chi_F(z) dA(z) \\ &\leq \left(\int_\Omega M_\Omega J_\lambda(z)^s g \circ \phi(z)^{2s} \chi_F(z) dA(z) \right)^{\frac{1}{s}} |F|^{1-\frac{1}{s}}. \end{aligned}$$

Now we use Lemma 4.1 to bound the last integral above by a constant times

$$\int_\Omega J(z, \phi)^{\lambda s} M(g \circ \phi^{2s} \chi_F)(z) dA(z).$$

For any $r > 1$, we can bound the above integral by

$$\left(\int_{\mathbb{C}} J(z, \phi)^{\lambda s} M(g \circ \phi^{2s} \chi_F)(z)^r dA(z) \right)^{\frac{1}{r}} \left(\int_\Omega J(z, \phi)^{\lambda s} dA(z) \right)^{1-\frac{1}{r}}.$$

The first inequality at (4.6) guarantees that the weight $J(\cdot, \phi)^{\lambda s}$ belongs to the Muckenhoupt class A_r (see [10] for details). Therefore we can use the weighted L^r inequality for the maximal function and a change of coordinates to see that

$$\begin{aligned} \int_{\mathbb{C}} J(z, \phi)^{\lambda s} M(g \circ \phi^{2s} \chi_F)(z)^r dA(z) &\leq C_r \int_F J(z, \phi)^{\lambda s} g \circ \phi(z)^{2sr} dA(z) \\ &= C_r \int_{\phi(F)} J(w, \phi^{-1})^{1-\lambda s} g(w)^{2sr} dA(w), \end{aligned}$$

where C_r is a positive constant depending on r . Summarizing, we get for the integral at (4.4) the bound

$$C |F|^{1-\frac{1}{s}} \left(\int_\Omega J(z, \phi)^{\lambda s} dA(z) \right)^{\frac{1}{s}-\frac{1}{sr}} \left(\int_{\phi(F)} J(w, \phi^{-1})^{1-\lambda s} g(w)^{2sr} dA(w) \right)^{\frac{1}{rs}}$$

For the second integral above, we only use Hölder's inequality again, and obtain the bound

$$\left(\int_{\phi(F)} g(w)^{p_\lambda} dA(w) \right)^{\frac{2rs}{p_\lambda}} \left(\int_{\phi(F)} J(w, \phi^{-1})^{\frac{p_\lambda(1-\lambda s)}{p_\lambda-2rs}} dA(w) \right)^{\frac{p_\lambda-2rs}{p_\lambda}}$$

which is finite provided that both $p_\lambda > 2rs$ and $\frac{p_\lambda(1-\lambda s)}{p_\lambda-2rs} < \frac{K}{K-1}$ hold. But both facts are guaranteed by our choice of parameters, in particular to the second inequality at (4.6). This means that the integral at (4.4) has the upper bound

$$\frac{C|F|^{1-\frac{1}{s}}}{\lambda-\beta} \left(\int_{\Omega} J(z, \phi)^{\lambda s} dA(z) \right)^{\frac{1}{s}-\frac{1}{sr}} \|g\|_{L^{p_\lambda}(\phi(F))}^2 \left(\int_{\phi(F)} J(w, \phi^{-1})^{\frac{p_\lambda(1-\lambda s)}{p_\lambda-2rs}} dA(w) \right)^{\frac{p_\lambda-2rs}{p_\lambda}}$$

Since both ϕ and ϕ^{-1} are normalized K -quasiconformal mappings, the two integrals above are bounded by constants depending only on K . One obtains for the integral at (4.3) the bound

$$\frac{C}{(\lambda-\beta)^{\frac{1}{2}}} \|g\|_{L^{p_\lambda}(\phi(F))} \leq \frac{C}{(\lambda-\beta)^{\frac{1}{2}}} \|\mu\|_{L^\infty(\mathbb{C})}^{1-\theta} \|\mu\|_{W^{\alpha,2}(\mathbb{C})}^\theta$$

where the constant C depends on $r, s, \lambda, \theta, \alpha$ and K .

To find larger possible β , we have to find the supremum of those λ for which the pair (θ, λ) belongs to the set

$$A = \left\{ (\theta, \lambda); 0 < \theta < 1, \beta < \lambda < \theta\alpha, \lambda + (1-\alpha)\theta < \frac{1}{Ks} \right\}.$$

according to (4.5). This supremum is easily seen to be $\frac{\alpha}{Ks}$. Further, all the above argument works for every $s \in (1, \frac{\alpha}{\beta K})$, so that the bound for (4.3) reads now as

$$\frac{C_K}{\frac{\alpha}{K} - \beta} \|\mu\|_{W^{\alpha,2}(\mathbb{C})}^{\frac{1}{K}}.$$

Summarizing,

$$\begin{aligned} & \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{|\mu(\phi(z+w)) - \mu(\phi(z))|^2}{|w|^{2+2\beta}} dA(z) dA(w) \\ & \leq C_K \left(\frac{\|\mu\|_{L^2(\mathbb{C})}^2}{\beta} + \frac{1}{\frac{\alpha}{K} - \beta} \|\mu\|_{W^{\alpha,2}(\mathbb{C})}^{\frac{2}{K}} \right) \end{aligned}$$

for all $\beta \in (0, \frac{\alpha}{K})$. Equivalently, we have an inequality for the nonhomogeneous norms

$$\|\mu \circ \phi\|_{W^{\beta,2}(\mathbb{C})} \leq C(\alpha, \beta, K) \|\mu\|_{W^{\alpha,2}(\mathbb{C})}^{\frac{1}{K}},$$

as stated. \square

Remark 4.3. The condition $\beta < \frac{\alpha}{K}$ is by no means sharp. This is clear when α is close to 1 but also it can be seen from the fact that we are using the Hölder regularity of ϕ . It seems that one can get a factor which runs between $1/K$ and 1 in terms of α . As promised in the introduction this will be a matter of a forthcoming work.

4.2 Regularity of homeomorphic solutions

We start by recalling the basic result on the existence of homeomorphic solutions to Beltrami type equations. In absence of extra regularity the integrability of the solutions comes from the work of Astala [7]. We recall the proof in terms of Neumann series since it will be used both in this section and in the sequel.

Lemma 4.4. *Let μ, ν be bounded functions, compactly supported in \mathbb{D} , such that $||\mu(z)| + |\nu(z)|| \leq \frac{K-1}{K+1}$ at almost every $z \in \mathbb{C}$. The equation*

$$\bar{\partial}f = \mu \partial f + \nu \bar{\partial}f \quad (4.7)$$

admits only one homeomorphic solution $\phi : \mathbb{C} \rightarrow \mathbb{C}$, such that $|\phi(z) - z| = \mathcal{O}(1/|z|)$ as $|z| \rightarrow \infty$. Further, if $p \in (2, \frac{2K}{K-1})$ then the quantity

$$\|\partial\phi - 1\|_{L^p(\mathbb{C})} + \|\bar{\partial}\phi\|_{L^p(\mathbb{C})}$$

is bounded by a constant $C = C(K, p)$ that depends only on K and p .

Proof. Put $\phi(z) = z + \mathcal{C}h(z)$, where h is defined by

$$(I - \mu T - \nu \bar{T})h = \mu + \nu.$$

and \mathcal{C} and T denote, respectively, Cauchy and Beurling transforms. Since T is an isometry in $L^2(\mathbb{C})$, one can construct such a function h as Neumann series

$$h = \sum_{n=0}^{\infty} (\mu T + \nu \bar{T})^n (\mu + \nu)$$

which obviously defines an $L^2(\mathbb{C})$ function. By Riesz-Thorin interpolation theorem,

$$\lim_{p \rightarrow 2} \|T\|_p = 1,$$

it then follows that $h \in L^p(\mathbb{C})$ for every $p > 2$ such that $\|T\|_p < \frac{K+1}{K-1}$. Hence, the Cauchy transform $\mathcal{C}h$ is Hölder continuous (with exponent $1 - \frac{2}{p}$). Further, since h is compactly supported, we get $|\phi(z) - z| = |\mathcal{C}h(z)| \leq \frac{C}{|z|}$, and in fact $\phi - z$ belongs to $W^{1,p}(\mathbb{C})$ for such values of p . A usual topological argument proves that ϕ is a homeomorphism. For the uniqueness, note that if we are given two solutions ϕ_1, ϕ_2 as in the statement then $\bar{\partial}(\phi_1 \circ \phi_2^{-1}) = 0$

so that $\phi_1 \circ \phi_2^{-1}(z) - z$ is holomorphic on \mathbb{C} and vanishes at infinity. In order to recover the precise range $(\frac{2K}{K+1}, \frac{2K}{K-1})$ obtained by Astala [7] we need a remarkable result from [10] which says that $I - \mu T - \nu \overline{T} : L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C})$ defines a bounded invertible operator for these values of p . Further, both its L^p norm and that of its inverse depend only on K and p . This implies that for every $p \in (\frac{2K}{K+1}, \frac{2K}{K-1})$ there is a constant $C = C(K, p)$ such that

$$\|h\|_{L^p(\mathbb{C})} \leq C_{K,p}.$$

The claim follows since $\partial\phi - 1 = Th$ and $\overline{\partial}\phi = h$. \square

Once we know about the existence of homeomorphic solutions, it is time to check their regularity when the coefficients belong to some fractional Sobolev space.

Theorem 4.5. *Let $\alpha \in (0, 1)$, and suppose that $\mu, \nu \in W^{\alpha,2}(\mathbb{C})$ are Beltrami coefficients, compactly supported in \mathbb{D} , such that*

$$||\mu(z)| + |\nu(z)|| \leq \frac{K-1}{K+1}.$$

at almost every $z \in \mathbb{D}$. Let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be the only homeomorphism satisfying

$$\overline{\partial}\phi = \mu \partial\phi + \nu \overline{\partial}\phi$$

and $\phi(z) - z = \mathcal{O}(1/z)$ as $|z| \rightarrow \infty$. Then, $\phi(z) - z$ belongs to $W^{1+\theta\alpha,2}(\mathbb{C})$ for every $\theta \in (0, \frac{1}{K})$, and

$$\|D^{1+\theta\alpha}(\phi - z)\|_{L^2(\mathbb{C})} \leq C_K \left(\|\mu\|_{W^{\alpha,2}(\mathbb{C})}^\theta + \|\nu\|_{W^{\alpha,2}(\mathbb{C})}^\theta \right)$$

for some constant C_K depending only on K .

Proof. We consider a \mathcal{C}^∞ function ψ , compactly supported inside of \mathbb{D} , such that $0 \leq \psi \leq 1$ and $\int \psi = 1$. For $n = 1, 2, \dots$ let $\psi_n(z) = n^2 \psi(nz)$. Put

$$\mu_n(z) = \int_{\mathbb{C}} \mu(w) \psi_n(z - w) dA(w),$$

and

$$\nu_n(z) = \int_{\mathbb{C}} \nu(w) \psi_n(z - w) dA(w).$$

It is clear that both μ_n, ν_n are compactly supported in $\frac{n+1}{n}\mathbb{D}$, $|\mu_n(z)| + |\nu_n(z)| \leq \frac{K-1}{K+1}$, $\|\mu_n - \mu\|_{W^{\alpha,2}(\mathbb{C})} \rightarrow 0$ and $\|\nu_n - \nu\|_{W^{\alpha,2}(\mathbb{C})} \rightarrow 0$ as $n \rightarrow \infty$. Indeed there is convergence in L^p for all $p \in (1, \infty)$. Thus, by interpolation we then get that for any $0 < \theta < 1$

$$\lim_{n \rightarrow \infty} \|\mu_n - \mu\|_{W^{\alpha\theta, \frac{2}{\theta}}(\mathbb{C})} + \|\nu_n - \nu\|_{W^{\alpha\theta, \frac{2}{\theta}}(\mathbb{C})} = 0$$

and in particular, the sequences $D^{\alpha\theta}\mu_n$ and $D^{\alpha\theta}\nu_n$ are bounded in $L^{\frac{2}{\theta}}(\mathbb{C})$. Let ϕ_n be the only K -quasiconformal mapping $\phi_n : \mathbb{C} \rightarrow \mathbb{C}$ satisfying

$$\bar{\partial}\phi_n = \mu_n \partial\phi_n + \nu_n \overline{\partial\phi_n} \quad (4.8)$$

and normalized by $\phi_n(z) - z = \mathcal{O}_n(1/z)$ as $|z| \rightarrow \infty$. By the construction in Lemma 4.4, $\phi_n(z) = z + \mathcal{C}h_n(z)$ where h_n is the only $L^2(\mathbb{C})$ solution to

$$h_n = \mu_n Th_n + \nu_n \overline{Th_n} + (\mu_n + \nu_n),$$

and $\mathcal{C}h_n$ denotes the Cauchy transform. As in Lemma 4.4, h_n belongs to $L^p(\mathbb{C})$ for all $p \in (\frac{2K}{K+1}, \frac{2K}{K-1})$ and $\|h_n\|_{L^p(\mathbb{C})} \leq C = C(K, p)$; in particular, $\phi_n - z$ is a bounded sequence in $W^{1,p}(\mathbb{C})$.

We now write equation (4.8) as

$$\bar{\partial}(\phi_n - z) = \mu_n \partial(\phi_n - z) + \nu_n \overline{\partial(\phi_n - z)} + \mu_n + \nu_n$$

and take fractional derivatives. If $\beta = \alpha\theta$, we can use Lemma 3.1 (a) to find two functions E_β, F_β such that

$$\begin{aligned} D^\beta \bar{\partial}(\phi_n - z) &= D^\beta \mu_n \partial(\phi_n - z) + \mu_n D^\beta \partial(\phi_n - z) + E_\beta \\ &\quad + D^\beta \nu_n \overline{\partial(\phi_n - z)} + \nu_n D^\beta \overline{\partial(\phi_n - z)} + F_\beta. \end{aligned}$$

Further, E_β satisfies

$$\|E_\beta\|_{L^2(\mathbb{C})} \leq C_0 \|D^\beta \mu\|_{L^{p_1}(\mathbb{C})} \|\partial(\phi_n - z)\|_{L^{p_2}(\mathbb{C})}, \quad (4.9)$$

where p_2 is any real number with $2 < p_2 < \frac{2K}{K-1}$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$, and C_0 depends on p_1, p_2 . Analogously,

$$\|F_\beta\|_{L^2(\mathbb{C})} \leq C_0 \|D^\beta \nu\|_{L^{p_1}(\mathbb{C})} \|\partial(\phi_n - z)\|_{L^{p_2}(\mathbb{C})}. \quad (4.10)$$

Now we notice that we have $D^\beta \partial\varphi = \partial D^\beta \varphi$ and similarly for $\bar{\partial}$. Further, if φ is real then $D^\beta \varphi$ is also real. Thus

$$\begin{aligned} \bar{\partial} D^\beta(\phi_n - z) &= \mu_n \partial D^\beta(\phi_n - z) + D^\beta \mu_n \partial(\phi_n - z) + E_\beta \\ &\quad + \nu_n \overline{\partial(D^\beta(\phi_n - z))} + D^\beta \nu_n \overline{\partial(\phi_n - z)} + F_\beta, \end{aligned}$$

or equivalently

$$\begin{aligned} (I - \mu_n T - \nu_n \overline{T})(\bar{\partial}) \left(D^\beta(\phi_n - z) \right) &= D^\beta \mu_n \partial(\phi_n - z) + D^\beta \nu_n \overline{\partial(\phi_n - z)} + E_\beta + F_\beta. \end{aligned}$$

The term on the right hand side is actually an $L^2(\mathbb{C})$ function. To see this, it suffices to choose in both (4.9) and (4.10) the value $p_1 = \frac{2}{\theta}$ for some $\theta \in (0, \frac{1}{K})$. Now, the operator $I - \mu_n T - \nu_n \bar{T}$ is continuously invertible in $L^2(\mathbb{C})$, and a Neumann series argument shows that the norm of its inverse is bounded by $\frac{1}{2}(K+1)$. Thus,

$$\begin{aligned} & \|\bar{\partial} D^\beta(\phi_n - z)\|_{L^2(\mathbb{C})} \\ & \leq C_0 \frac{K+1}{2} \left(\|D^\beta \mu_n\|_{L^{\frac{2}{\theta}}(\mathbb{C})} + \|D^\beta \nu_n\|_{L^{\frac{2}{\theta}}(\mathbb{C})} \right) \|\partial(\phi_n - z)\|_{L^{p_2}(\mathbb{C})} \\ & \leq C_0 \frac{K+1}{2} \left(\frac{K-1}{K+1} \right)^{1-\theta} \left(\|\mu_n\|_{W^{\alpha,2}(\mathbb{C})}^\theta + \|\nu_n\|_{W^{\alpha,2}(\mathbb{C})}^\theta \right) \|\partial(\phi_n - z)\|_{L^{p_2}(\mathbb{C})} \\ & \leq C_0 \frac{K+1}{2} \left(\|\mu_n\|_{W^{\alpha,2}(\mathbb{C})}^\theta + \|\nu_n\|_{W^{\alpha,2}(\mathbb{C})}^\theta \right) \|\partial(\phi_n - z)\|_{L^{p_2}(\mathbb{C})} \end{aligned}$$

where C_0 is the constant in (4.9). As $n \rightarrow \infty$, the right hand side is bounded by

$$C_0 \frac{K+1}{2} \left(\|\mu\|_{W^{\alpha,2}(\mathbb{C})}^\theta + \|\nu\|_{W^{\alpha,2}(\mathbb{C})}^\theta \right) C(K, p_2)$$

because $\|\partial(\phi - z)\|_{L^{p_2}(\mathbb{C})} \leq C(K, p_2)$. Hence, $\bar{\partial} D^\beta(\phi_n - z)$ is bounded in $L^2(\mathbb{C})$, and thus also $\partial D^\beta(\phi_n - z)$, because T is an isometry of $L^2(\mathbb{C})$ and $T(\bar{\partial} D^\beta(\phi_n - z)) = \partial D^\beta(\phi_n - z)$. Therefore, by passing to a subsequence we see that $D^\beta(\phi_n - z)$ converges in $W^{1,2}(\mathbb{C})$, and as a consequence $\phi - z$ belongs to $W^{1+\beta,2}(\mathbb{C})$. Further, we have the bounds

$$\|D^{1+\theta\alpha}(\phi - z)\|_{L^2(\mathbb{C})} \leq C \left(\|\mu\|_{W^{\alpha,2}(\mathbb{C})}^\theta + \|\nu\|_{W^{\alpha,2}(\mathbb{C})}^\theta \right)$$

for some constant C depending only on K . \square

4.3 Regularity of complex geometric optics solutions

We are now ready to give precise bounds on the Sobolev regularity of the complex geometric optics solutions to the equation $\bar{\partial} f = \mu \bar{\partial} \bar{f}$ introduced in Theorem 2.2.

Theorem 4.6. *Let $\mu \in W^{\alpha,2}(\mathbb{C})$ be a Beltrami coefficient, compactly supported in \mathbb{D} , with $\|\mu\|_\infty \leq \frac{K-1}{K+1}$ and $\|\mu\|_{W^{\alpha,2}(\mathbb{C})} \leq \Gamma_0$. Let $f = f_\mu(z, k)$ the complex geometric optics solutions to the equation*

$$\bar{\partial} f = \mu \bar{\partial} \bar{f}.$$

For any $0 < \theta < \frac{1}{K}$ we have that

$$f \in W_{loc}^{1+\theta\alpha,2}(\mathbb{C}).$$

Further, we have the estimate

$$\|D^{1+\alpha\theta}(f_\mu)(\cdot, k)\|_{L^2(\mathbb{D})} \leq C(K) e^{C|k|} (1 + |k|) (\Gamma_0 + |k|^\alpha)^\theta$$

where $C, C(K) > 0$, and $C(K)$ depends only on K .

Proof. The existence and uniqueness of the complex geometric optics solutions comes from [11, Theorem 4.2] (see Theorem 2.2 in the present paper). Secondly, it is shown in [11, Lemma 7.1] that f may be represented as

$$f = e^{ik\phi}$$

where $\phi : \mathbb{C} \rightarrow \mathbb{C}$ is the only $W_{loc}^{1,2}(\mathbb{C})$ homeomorphism solving

$$\bar{\partial}\phi = -\mu \frac{\bar{k}}{k} e_{-k}(\phi) \bar{\partial}\bar{\phi} \quad (4.11)$$

and normalized by the condition $|\phi(z) - z| \rightarrow 0$ as $|z| \rightarrow \infty$. Here $e_{-k}(\phi(z)) = e^{-ik\phi(z) - i\bar{k}\overline{\phi(z)}}$ is a unimodular function so that $|e_{-k}(\phi(z))| = 1$. In particular, ϕ is conformal outside of \mathbb{D} , because $\text{supp}(\mu) \subset \mathbb{D}$. Thus by Koebe $\frac{1}{4}$ theorem

$$\phi(\mathbb{D}) \subset 4\mathbb{D} \Rightarrow \|\phi(\cdot, k)\|_{L^\infty(\mathbb{D})} \leq 4 \quad (4.12)$$

Our first task is to determine the Sobolev regularity of the coefficient $\mu e_{-k}(\phi)$ in equation (4.11). We will argue by interpolation. Firstly, by (4.12),

$$\|e_{-k}(\phi)\|_{L^2(\mathbb{D})} \leq |\mathbb{D}|^{\frac{1}{2}}.$$

For the L^2 norm of the derivative, we invoke Lemma 4.4 to obtain that

$$\|D(\phi - z)\|_{L^2(\mathbb{C})} \leq C(K),$$

Thus,

$$\|D(e_{-k}(\phi))\|_{L^2(\mathbb{D})} \leq C(K) |k| |\mathbb{D}|^{\frac{1}{2}}.$$

and by interpolation we arrive to,

$$\|e_{-k}(\phi)\|_{W^{\alpha,2}(\mathbb{D})} \leq C(K) |\mathbb{D}|^{\frac{1}{2}} |k|^\alpha.$$

Now we will use the remark 3.2 to see that that $\mu e_{-k}(\phi)$ belongs also to $W^{\alpha,2}(\mathbb{C})$. Since $e_{-k}(\phi)$ is unimodular, the L^2 bound is obvious. By virtue of (3.4) we have that

$$\begin{aligned} \|D^\alpha(\mu e_{-k}(\phi))\|_{L^2(\mathbb{C})} &\leq C(\|D^\alpha \mu\|_{L^2(\mathbb{C})} \|e_{-k}(\phi)\| + \kappa \|D^\alpha(e_{-k}(\phi))\|_{L^2(\mathbb{D})}) \\ &\leq C |\mathbb{D}|^{\frac{1}{2}} (|k|^\alpha + \Gamma_0) \end{aligned} \quad (4.13)$$

The bound (4.13) allows us to apply Theorem 4.5 to equation (4.11). We obtain that $\phi_0(z) = \phi(z) - z$ satisfies the estimate

$$\|D^{\alpha\theta}(\partial\phi_0)\|_{L^2(\mathbb{C})} + \|D^{\alpha\theta}(\bar{\partial}\phi_0)\|_{L^2(\mathbb{C})} \leq C_K (\Gamma_0 + |k|^\alpha)^\theta \quad (4.14)$$

for $\theta \in (0, \frac{1}{K})$. We push this bound to f . Since $f(z) = e^{ik\phi(z)}$, we have

$$\partial f(z) = e^{ik\phi(z)} ik \partial\phi(z)$$

and again from Lemma 3.1 (c), for any disk D ,

$$\begin{aligned} \|D^{\alpha\theta}(\partial f)\|_{L^2(D)} &\leq \|D^{\alpha\theta}(e^{ik\phi}) ik\partial\phi\|_{L^2(D)} + \|ik D^{\alpha\theta}(\partial\phi) e^{ik\phi}\|_{L^2(D)} \\ &\quad + C \|e^{ik\phi}\|_{L^\infty(D)} \|D^{\alpha\theta}(\partial\phi)\|_{L^2(D)}. \end{aligned} \quad (4.15)$$

For the second and third terms on the right hand side above, we notice that (4.12) yields that

$$\sup_{z \in \mathbb{D}} |e^{ik\phi(z)}| \leq e^{4|k|}, \quad (4.16)$$

which combined with (4.14) provides us with the estimate

$$\begin{aligned} \|ik D^{\alpha\theta}(\partial\phi) e^{ik\phi}\|_{L^2(\mathbb{D})} &\leq |k| e^{C|k|} \|D^{\alpha\theta}(\partial\phi)\|_{L^2(\mathbb{D})} \\ &\leq |k| e^{C|k|} C_K (\Gamma_0 + |k|^\alpha)^\theta. \end{aligned} \quad (4.17)$$

Concerning the first term in (4.15), we recall that L^p bounds for $\partial\phi$ are possible only for $p \in (\frac{2K}{K+1}, \frac{2K}{K-1})$. This forces us to look for L^q bounds for $D^{\alpha\theta}(e^{ik\phi})$, for some $q > 2K$. These bounds are easily obtained by interpolation. More precisely, we know the L^∞ bound given at (4.12). Further, we have also a $W^{1,2}$ bound,

$$\int_{\mathbb{D}} |\partial(e^{ik\phi(z)})|^2 dA(z) \leq |k|^2 e^{C|k|} K \int_{\mathbb{D}} J(z, \phi) dA(z) \leq |k|^2 e^{C|k|} K |\mathbb{D}|.$$

Thus, by interpolation we obtain

$$\begin{aligned} \|D^{\alpha\theta}(e^{ik\phi})\|_{L^{\frac{2}{\alpha\theta}}(\mathbb{D})} &\leq C \|e^{ik\phi}\|_{L^\infty(\mathbb{D})}^{1-\alpha\theta} \|\partial(e^{ik\phi})\|_{L^2(\mathbb{D})}^{\alpha\theta} \\ &\leq |k|^{\alpha\theta} e^{C|k|} (K|\mathbb{D}|)^{\frac{\alpha\theta}{2}} \leq C(K) |k|^{\alpha\theta} e^{C|k|}. \end{aligned}$$

Now recall that $0 < \theta < \frac{1}{K}$ is fixed, and let $p = \frac{2}{\alpha\theta}$. If we now consider any real number s such that $\frac{2}{1-\alpha\theta} < s < \frac{2K}{K-1}$, we obtain

$$\begin{aligned} \|D^{\alpha\theta}(e^{ik\phi}) ik\partial\phi\|_{L^2(\mathbb{D})} &\leq |k| \|D^{\alpha\theta}(e^{ik\phi})\|_{L^{\frac{2}{\alpha\theta}}(\mathbb{D})} \|\partial\phi\|_{L^{\frac{2}{1-\alpha\theta}}(\mathbb{D})} \\ &\leq C(K) |k|^{1+\alpha\theta} e^{C|k|} \|\partial\phi\|_{L^s(\mathbb{D})} \\ &= C(K) |k|^{1+\alpha\theta} e^{C|k|} \end{aligned}$$

because the normalization on ϕ forces uniform bounds for $\|\partial\phi\|_{L^s(\mathbb{D})}$ depending only on K . Summarizing, (4.15) gives us the bound

$$\|D^{\alpha\theta}(\partial f)\|_{L^2(\mathbb{D})} \leq C(K) e^{C|k|} (1 + |k|) (\Gamma_0 + |k|^\alpha)^\theta.$$

Similar calculations give the corresponding bound for $D^{\alpha\theta}(\bar{\partial}f)$. \square

We will also need the following bounds in Section 6.

Lemma 4.7. *Let μ be a Beltrami coefficient, compactly supported in \mathbb{D} . Assume that $\|\mu\|_\infty \leq \frac{K-1}{K+1}$ and $\|\mu\|_{W^{\alpha,2}(\mathbb{C})} \leq \Gamma_0$. Let $f = f_\mu(z, k)$ denote the complex geometric optics solutions to*

$$\bar{\partial}f = \mu \bar{\partial}f.$$

Let $p < 2/(K-1)$. Then, for any disk D

$$\int_D \left| \frac{1}{\bar{\partial}f(z)} \right|^p \leq C$$

where the constant C depends on $\text{diam}(D)$, k , K and Γ_0 .

Proof. The function f can be represented as

$$f = e^{ik\phi}$$

so that

$$\frac{1}{\bar{\partial}f} = \frac{1}{e^{ik\phi}} \frac{1}{ik\bar{\partial}\phi}.$$

As we have seen in (4.16) in the proof of the above Theorem gives also lower local uniform bounds for $e^{ik\phi}$. Thus, only L^p bounds for $\bar{\partial}\phi$ are needed. But these bounds come from the fact that ϕ is a normalized K -quasiconformal mapping, so that

$$\bar{\partial}\phi \in L^p$$

□

5 Uniform subexponential decay

We investigate the decay property of complex geometric optic solutions to the equation

$$\bar{\partial}f_\lambda = \lambda\mu \bar{\partial}f_\lambda,$$

where $\lambda \in \partial\mathbb{D}$ is a fixed complex parameter, and $\mu \in W_0^{\alpha,2}(\mathbb{C})$ is a Beltrami coefficient compactly supported in \mathbb{D} . It turns out that f_λ admits the representation

$$f_\lambda(z, k) = e^{ik\phi_\lambda(z, k)}$$

where ϕ_λ satisfies the following properties (see [11, Lemma 7.1] or the proof of Theorem 4.6 above):

1. $\phi_\lambda(\cdot, k) : \mathbb{C} \rightarrow \mathbb{C}$ is a quasiconformal mapping.
2. $\phi_\lambda(z, k) = z + \mathcal{O}_k(1/z)$ as $|z| \rightarrow \infty$
3. ϕ_λ satisfies the *nonlinear* equation

$$\bar{\partial}\phi_\lambda(z, k) = -\lambda\mu(z) \frac{\bar{k}}{k} e_{-k}(\phi_\lambda(z, k)) \overline{\bar{\partial}\phi_\lambda(z, k)} \quad (5.1)$$

As was explained in Section 2, our goal is to obtain a uniform decay of the type

$$|\phi_\lambda(z, k) - z| \leq \frac{C}{|k|^{b\alpha}} \quad (5.2)$$

The precise statement can be found at Theorem 5.7. For the proof, we will mainly follow the lines of both [11, 15]. This consists on investigating first the behaviour of *linear* Beltrami equations with the rapidly oscillating coefficients $\mu(z) e_{-k}(z)$, and then treat the nonlinearity as a perturbation.

5.1 Estimates for the linear equation

As usually, μ denotes a Beltrami coefficient, compactly supported in \mathbb{D} , with the ellipticity bound

$$\|\mu\|_{L^\infty(\Omega)} \leq \frac{K-1}{K+1} = \kappa$$

and the smoothness assumption

$$\|\mu\|_{W^{\alpha,2}(\mathbb{C})} = \|\mu\|_{L^2(\mathbb{C})} + \|D^\alpha \mu\|_{L^2(\mathbb{C})} \leq \Gamma_0$$

for some $0 < \alpha < 1$ and $\Gamma_0 > 0$. For each complex numbers $k \in \mathbb{C}$ and $\lambda \in \mathbb{D}$, let $\psi = \psi_\lambda(z, k)$ be the only homeomorphic solution to the problem,

$$\begin{cases} \bar{\partial}\psi(z, k) = \frac{\bar{k}}{k} \lambda e_{-k}(z) \mu(z) \partial\psi(z, k) \\ \psi(z, k) - z = \mathcal{O}(1/z), \quad z \rightarrow \infty \end{cases} \quad (5.3)$$

Then ψ can be represented by means of a Cauchy transform

$$\psi(z, k) - z = \int_{\mathbb{C}} \bar{\partial}\psi(w, k) \Phi(z, w) dA(w), \quad (5.4)$$

where $\Phi(z, w) = \frac{\psi_{\mathbb{D}}(w)}{z-w}$ for a smooth cutoff function $\psi_{\mathbb{D}} = 1$ on \mathbb{D} (in particular on the support of $\bar{\partial}\psi$). We need subtle properties for both terms.

Lemma 5.1. *Let n_0 be given, and let $s \geq 2$ be such that*

$$\kappa \|T\|_s < 1.$$

There exists a decomposition $\bar{\partial}\psi_\lambda(z, k) = g_\lambda(z, k) + h_\lambda(z, k)$ satisfying the following properties:

1. $\|h_\lambda(\cdot, k)\|_s \leq C(\kappa, s) (\kappa \|T\|_s)^{n_0}$.
2. $\|g_\lambda(\cdot, k)\|_s \leq C(\kappa)$.

3. For $R > 0$ and $|k| > 2R$,

$$\left(\int_{|\xi| < R} |\widehat{g}_\lambda(\xi, k)|^q dA(\xi) \right)^{\frac{1}{q}} \leq C(\alpha, \kappa, p) M(p)^{n_0} \frac{\Gamma_0}{|k|^\alpha}$$

where $1 < p < 2$, $q = \frac{p}{p-1}$, $\widehat{g}_\lambda(\xi, k) = (g_\lambda(\cdot, k))^\wedge(\xi)$ and the value of $M(p)$ is given in (5.8).

The proof will rely on the Neumann series expression of $\bar{\partial}\psi$. For this, we consider the unimodular factors

$$e_n(z) = e^{in(kz + i\bar{k}\bar{z})}.$$

An idea which goes back to [11] is dealing with the unimodular factors e_n conjugating them with the Beurling transform. Namely, we express

$$\bar{\partial}\psi = \sum_n \left(\frac{\bar{k}}{k} \lambda \right)^{n+1} e_{-(n+1)} f_n \quad (5.5)$$

where

$$\begin{cases} f_0 = \mu \\ f_n = \mu T_n(f_{n-1}), n = 1, 2, \dots \end{cases} \quad (5.6)$$

Here by T_n we denote a singular integral operator defined by the rule

$$T_n(\varphi) = e_n T(e_{-n}\varphi)$$

where T is the usual Beurling transform (4.2). It is not hard to see that T_n is represented, at the frequency side, by a unimodular multiplier of the form

$$\widehat{T_n\varphi}(\xi) = \frac{\xi - n}{\xi + n} \widehat{\varphi}(\xi)$$

Thus,

$$\|T_n\|_{L^2(\mathbb{C})} = \|T_n\|_{L^2(\mathbb{C}) \rightarrow L^2(\mathbb{C})} = 1$$

and T_n is an isometry of $L^2(\mathbb{C})$. In fact, for any $1 < p < \infty$,

$$\|T_n(\varphi)\|_{L^p(\mathbb{C})} = \|T(e_{-n}\varphi)\|_{L^p(\mathbb{C})} \leq \|T\|_{L^p(\mathbb{C})} \|\varphi\|_{L^p(\mathbb{C})}$$

because $|e_n(z)| = 1$, so that $\|T_n\|_{L^p(\mathbb{C})} = \|T\|_{L^p(\mathbb{C})}$. As T_n is given by a Fourier multiplier, it commutes with any constant coefficients differential operator D and thus,

$$\|T_n\varphi\|_{W^{1,p}(\mathbb{C})} = \|T_n\varphi\|_{L^p(\mathbb{C})} + \|T_n(D\varphi)\|_{L^p(\mathbb{C})} \leq \|T\|_p \|\varphi\|_{W^{1,p}(\mathbb{C})}$$

and therefore $\|T_n\|_{W^{1,p}(\mathbb{C})} \leq \|T\|_{L^p(\mathbb{C})}$. Furthermore, the complex interpolation method gives that for any $0 < \beta < 1$,

$$\|T_n\|_{W^{\beta,p}(\mathbb{C})} \leq C_0 \|T\|_{L^p(\mathbb{C})} \quad (5.7)$$

where $C_0 > 0$ is a universal constant.

Let $1 < p < 2$ be fixed. We declare

$$\begin{aligned} B &= B(p) = \|T_n\|_{W^{\alpha,p}(\mathbb{C})} \leq \|T\|_{L^p(\mathbb{C})} \\ D &= D(p) = \|T_n\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} \leq \|T\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} \\ M &= M(p) = B + D \end{aligned} \quad (5.8)$$

It is well known that the series $\sum_n f_n$ defines a compactly supported $L^2(\mathbb{C})$ function (actually $L^p(\mathbb{C})$ for any $1 + \kappa < p < 1 + \frac{1}{\kappa}$). Next lemma yields Sobolev estimates for f_n in terms of the Sobolev norm $\|\mu\|_{W^{\alpha,2}(\mathbb{C})}$.

Lemma 5.2. *For any $1 < p < 2$ there exists a constant $C = C(p)$ such that*

$$\|f_n\|_{W^{\alpha,p}(\mathbb{C})} \leq C(p) \Gamma_0 \kappa^{n-1} (M(p))^n,$$

for any $n = 1, 2, \dots$

Proof. To prove the Lemma, we will use Remark 3.2 for the complex dilatation μ and an arbitrary function $g \in W^{\alpha,p}$. Then it holds that

$$\|D^\alpha(\mu g)\|_{L^p(\mathbb{C})} \leq \kappa \|D^\alpha g\|_{L^p(\mathbb{D})} + C \Gamma_0 \|g\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} \quad (5.9)$$

for some positive constant $C = C(p) \geq 1$. First of all, we study the L^p norm of f_n . Recalling that μ is compactly supported inside of \mathbb{D} , we first see that

$$\|f_n\|_{L^p(\mathbb{C})} = \|f_n\|_{L^p(\mathbb{D})} \leq \kappa \|T_n f_{n-1}\|_{L^p(\mathbb{D})}.$$

Next, (5.9) yields that,

$$\begin{aligned} \|D^\alpha f_n\|_{L^p(\mathbb{C})} &= \|D^\alpha(\mu T_n f_{n-1})\|_{L^p(\mathbb{C})} \\ &\leq C \Gamma_0 \|T_n f_{n-1}\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} + \kappa \|D^\alpha T_n f_{n-1}\|_{L^p(\mathbb{D})} \end{aligned}$$

Hence, for any $n > 1$,

$$\begin{aligned} \|f_n\|_{W^{\alpha,p}(\mathbb{C})} &= \|f_n\|_{L^p(\mathbb{C})} + \|D^\alpha f_n\|_{L^p(\mathbb{C})} \\ &\leq C \Gamma_0 \|T_n f_{n-1}\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} + \kappa \|T_n f_{n-1}\|_{W^{\alpha,p}(\mathbb{C})} \end{aligned}$$

To control the first term above, we see that

$$\begin{aligned} \|T_n f_{n-1}\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} &\leq D \|f_{n-1}\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} \leq (D\kappa) \|T_{n-1} f_{n-2}\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} \\ &\leq (D\kappa)^{n-1} \|T_1 f_0\|_{L^{\frac{2p}{2-p}}(\mathbb{C})} \leq (D\kappa)^n |\mathbb{D}|^{\frac{1}{p}-\frac{1}{2}} \end{aligned}$$

and for the second, if $n > 1$

$$\|T_n f_{n-1}\|_{W^{\alpha,p}(\mathbb{C})} \leq B \|f_{n-1}\|_{W^{\alpha,p}(\mathbb{C})}.$$

If we denote $X_n = \|f_n\|_{W^{\alpha,p}(\mathbb{C})}$ then we have just seen that

$$X_n \leq C_1 (\kappa D)^n + (\kappa B) X_{n-1} \quad (5.10)$$

whenever $n > 1$, and where $C_1 = C \Gamma_0 |\mathbb{D}|^{\frac{1}{p}-\frac{1}{2}}$. For $n = 1$ we proceed differently. Since both $T_1 f_0$ and $D^\alpha T_1 f_0$ belong to $L^2(\mathbb{C})$, we can use Hölder's inequality to get

$$\begin{aligned} \|T_1 f_0\|_{L^p(\mathbb{D})} + \|D^\alpha T_1 f_0\|_{L^p(\mathbb{D})} &\leq (\|T_1 f_0\|_{L^2(\mathbb{C})} + \|D^\alpha T_1 f_0\|_{L^2(\mathbb{C})}) |\mathbb{D}|^{\frac{1}{p}-\frac{1}{2}} \\ &= \|T_1 f_0\|_{W^{\alpha,2}(\mathbb{C})} |\mathbb{D}|^{\frac{1}{p}-\frac{1}{2}} \\ &\leq |\mathbb{D}|^{\frac{1}{p}-\frac{1}{2}} B \|f_0\|_{W^{\alpha,2}(\mathbb{C})} = |\mathbb{D}|^{\frac{1}{p}-\frac{1}{2}} B \Gamma_0. \end{aligned}$$

Thus

$$X_1 \leq C_1 \kappa D + B |\mathbb{D}|^{\frac{1}{p}-\frac{1}{2}} \Gamma_0$$

Thus, after recursively using (5.10), we end up with

$$X_n \leq C_1 \kappa^n \sum_{j=0}^{n-1} B^j D^{n-j} + (\kappa B)^n \frac{\Gamma_0}{\kappa} |\mathbb{D}|^{\frac{1}{p}-\frac{1}{2}} \leq \tilde{C}_1 \kappa^{n-1} (B\kappa + D)^n$$

where $\tilde{C}_1 = \max\{C_1, \Gamma_0 |\mathbb{D}|^{\frac{1}{p}-\frac{1}{2}}\}$. Note finally that

$$\tilde{C}_1 \leq C(p) \Gamma_0,$$

which yields the claim. \square

In particular, every function f_n of the Neumann series is compactly supported and belongs to $L^p(\mathbb{C})$ for any $p \in (1, \infty)$, and also to $W^{\alpha,p}(\mathbb{C})$ for any $p < 2$.

Lemma 5.3. *If h belongs to $W^{\alpha,p}(\mathbb{C})$ for some $1 < p < 2$, then*

$$\left(\int_{|\xi|>R} |\widehat{h}(\xi)|^q dA(\xi) \right)^{\frac{1}{q}} \leq C(p) \frac{\|h\|_{W^{\alpha,p}(\mathbb{C})}}{R^\alpha}$$

Proof. We will use the characterization in terms of Bessel potentials of $W^{\alpha,p}(\mathbb{C})$. Since the Fourier transform maps continuously $L^p(\mathbb{C})$ into $L^q(\mathbb{C})$, we get that

$$\left(\int_{\mathbb{C}} \left((1 + |\xi|^2)^{\frac{\alpha}{2}} |\widehat{h}(\xi)| \right)^q dA(\xi) \right)^{\frac{1}{q}} \leq C(p) \|h\|_{\alpha,p}$$

Thus, a simple computation yields

$$\begin{aligned}
\left(\int_{|\xi|>R} |\widehat{h}(\xi)|^q dA(\xi) \right)^{\frac{1}{q}} &\leq \left(\int_{|\xi|>R} \left(\frac{(1+|\xi|^2)^{\frac{\alpha}{2}}}{|\xi|^\alpha} \right)^q |\widehat{h}(\xi)|^q dA(\xi) \right)^{\frac{1}{q}} \\
&\leq \frac{1}{R^\alpha} \left(\int_{\mathbb{C}} (1+|\xi|^2)^{\frac{\alpha q}{2}} |\widehat{h}(\xi)|^q dA(\xi) \right)^{\frac{1}{q}} \\
&\leq C(p) \frac{\|h\|_{\alpha,p}}{R^\alpha}
\end{aligned}$$

and the result follows. \square

Proof of Lemma 5.1. We use the Neumann series

$$\bar{\partial}\psi = \sum_n \left(\frac{\bar{k}}{k} \lambda \right)^{n+1} e_{-(n+1)k} f_n \quad (5.11)$$

introduced before. Then, take $g = \sum_{n=0}^{n_0} \left(\frac{\bar{k}}{k} \lambda e_{-k} \mu T \right)^n \left(\frac{\bar{k}}{k} \lambda e_{-k} \mu \right)$ and $h = \partial_{\bar{z}}\psi - g$. In this way, properties 1 and 2 follow easily from the general theory of the Beltrami equation, since

$$\begin{aligned}
&\left\| \left(\frac{\bar{k}}{k} \lambda e_{-k} \mu T \right)^n \left(\frac{\bar{k}}{k} \lambda e_{-k} \mu \right) \right\|_s \\
&\leq \kappa \|T\|_s \left\| \left(\frac{\bar{k}}{k} \lambda e_{-k} \mu T \right)^{n-1} \left(\frac{\bar{k}}{k} \lambda e_{-k} \mu \right) \right\|_s \\
&\leq (\kappa \|T\|_s)^n \|\mu\|_s = (\kappa \|T\|_s)^n \kappa |\mathbb{D}|^{\frac{1}{s}}.
\end{aligned}$$

For the proof of 3, we must use the regularity of μ . Use 5.11 to write $g(z, k) = \sum_{n=0}^{n_0} G_n(k, z)$ where $G_n(z, k) = \left(\frac{\bar{k}}{k} \lambda \right)^{n+1} e_{-(n+1)k} f_n$. Then, Lemma 5.2 can be applied to f_n . The Fourier transform of $G_n(z, k)$ (with respect to the z variable) reads as

$$\widehat{G_n}(\xi, k) = \left(\frac{\bar{k}}{k} \lambda \right)^{n+1} \widehat{f_n}(\xi - (n+1)k)$$

Hence, for $|k| > R$, we can use lemma 5.3, to get

$$\begin{aligned}
\left(\int_{|\xi| < R} |\widehat{g}(\xi, k)|^q dA(\xi) \right)^{\frac{1}{q}} &\leq \sum_{n=0}^{n_0} \left(\int_{|\xi| < R} |\widehat{G}_n(\xi, k)|^q dA(\xi) \right)^{\frac{1}{q}} \\
&= \sum_{n=0}^{n_0} \left(\int_{|\xi| < R} |\widehat{f}_n(\xi - (n+1)k)|^q dA(\xi) \right)^{\frac{1}{q}} \\
&= \sum_{n=0}^{n_0} \left(\int_{|\zeta + (n+1)k| < R} |\widehat{f}_n(\zeta)|^q dA(\zeta) \right)^{\frac{1}{q}} \\
&\leq \sum_{n=0}^{n_0} \left(\int_{|\zeta| > (n+1)|k| - R} |\widehat{f}_n(\zeta)|^q dA(\zeta) \right)^{\frac{1}{q}} \\
&\leq C(p) \sum_{n=0}^{n_0} \frac{\|f_n\|_{\alpha, p}}{((n+1)|k| - R)^\alpha}
\end{aligned}$$

where $C(p)$ is the constant from Lemma 5.3. Now, using Lemma 5.2,

$$\begin{aligned}
\left(\int_{|\xi| < R} |\widehat{g}(\xi, k)|^q dA(\xi) \right)^{\frac{1}{q}} &\leq C(\kappa, p) \frac{\Gamma_0}{\kappa} \sum_{n=0}^{n_0} \frac{(\kappa M(p))^n}{((n+1)|k| - R)^\alpha} \\
&\leq C(\kappa, p) (\kappa M(p))^{n_0} \frac{\Gamma_0}{\kappa} \sum_{n=0}^{n_0} \frac{1}{((n+1)|k| - R)^\alpha}
\end{aligned}$$

and if we take $|k| \geq 2R$, then we finally get

$$\begin{aligned}
\left(\int_{|\xi| < R} |\widehat{g}(\xi, k)|^q dA(\xi) \right)^{\frac{1}{q}} &\leq C(\alpha, \kappa, p) (\kappa M(p))^{n_0} \frac{\Gamma_0}{\kappa} \frac{1}{|k|^\alpha} \sum_{n=0}^{n_0} \frac{1}{(n + \frac{1}{2})^\alpha} \\
&\leq C(\alpha, \kappa, p) M(p)^{n_0} \frac{\Gamma_0}{\kappa} \frac{1}{|k|^\alpha} \\
&\leq C(\alpha, \kappa, p) M(p)^{n_0} \frac{\Gamma_0}{|k|^\alpha}
\end{aligned}$$

and the result follows. \square

The Cauchy kernel is not in L^2 but it belongs locally to $W^{\epsilon, p}$ for $1 < p < 2$, $\epsilon < \frac{2-p}{p}$. Thus we can work with a mollification of it which is perfectly controlled. However we need to choose carefully the mollification kernel (see [52] vol 1 & V.1).

Lemma 5.4. *There exists a $C_* > 0$ such that for any $N > 0$, there exists a C^∞ function ϕ_N in \mathbb{C} having the following properties:*

- $0 \leq \phi_N \leq 1$, $\phi_N = 1$ on \mathbb{D} and $\phi_N = 0$ on $2\mathbb{D}$.

- $\int \phi_N = 1$.
- $|D^\alpha \phi_N| \leq (C_* N)^{|\alpha|}$ for any $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \leq N$.

Lemma 5.5. *Let $\Phi(z, w) = \frac{\psi_{\mathbb{D}}}{z-w}$ and $1 < p < 2$.*

- (a) $\|\Phi(\cdot, z)\|_{L^p(\mathbb{D})} \leq C(p)$ for all $z \in \mathbb{C}$.
- (b) $\Phi(\cdot, z) \in W^{\epsilon, p}$ for $\epsilon < \frac{2-p}{p}$ uniformly in z .
- (c) For any $N > 0$, there exists a mollification $\Phi_{\delta, N}$ such that

$$\|\Phi(\cdot, z) - \Phi_{\delta, N}(\cdot, z)\|_{L^p(\mathbb{D})} \leq C(\epsilon, p) \delta^\epsilon$$

whenever $z \in \mathbb{C}$ and $\epsilon < \frac{2-p}{p}$.

- (d) $\|\Phi_{\delta, N}\|_{L^2(\mathbb{C})}$ blows up as a power of δ , i.e.

$$\|\Phi_{\delta, N}(\cdot, z)\|_{L^2(\mathbb{C})} \leq C(p) \delta^{1-\frac{2}{p}}$$

- (e) For each $R > \frac{1}{\delta}$ and $m > 0$, there exists a universal constant C_* and $C = C(p)$ such that for any $m \leq N$

$$\|\widehat{\Phi_{\delta, N}(\cdot, z)}\|_{L^2(|\xi| \geq R)} \leq C(p)(C_* N)^m \delta^{1-\frac{2}{p}} (\delta R)^{-m}$$

Proof. Claims (a) and (b) follow by the compactness of the support and Lemma 3.1. Now define

$$\widehat{\Phi_{\delta, N}(z, \cdot)}(\xi) = \widehat{\phi_N}(\delta \xi) \widehat{\Phi(z, \cdot)}(\xi).$$

Claim (c) follows from the fact that since $p < 2$, $W^{\epsilon, p} \subset B_e^{p, 2}$ (3.2). Namely,

$$\begin{aligned} \|\Phi_z(\cdot) - \Phi_{\delta, N}(z, \cdot)\|_{L^p} &\leq \int_{\mathbb{C}} \omega_p(\Phi_z)(w) \phi_\delta(w) dw \\ &\leq \|\Phi_z\|_{B_{\epsilon, p, 2}} \int (\phi_\delta(w))^2 |w|^{2+\epsilon 2} \frac{1}{2} \leq \delta^\epsilon \left(\int \phi^2(y) |y|^{2+\epsilon 2} \right)^{\frac{1}{2}} \leq \delta^\epsilon \|\phi\|_{L^2(\mathbb{C})} \end{aligned}$$

For claim (d), using Plancherel, Hölder, Hausdorff-Young inequalities and (a), we obtain, for $1/p - 1/q = 1/2$, that

$$\|\Phi_{\delta, N}\|_{L^2} \leq \|\Phi_z\|_{L^p} \|\widehat{\phi_N}(\delta \cdot)\|_{L^q} \leq C \delta^{1-\frac{2}{p}}.$$

For the last claim, write again

$$\begin{aligned} \|\widehat{\Phi_{\delta, N}}\|_{L^2(|\xi| > R_0)} &\leq \|\Phi_z\|_{L^p} \|\widehat{\phi_N}(\delta \xi)\|_{L^q(|\xi| > R_0)} \\ &\leq \|\Phi_z\|_{L^p} \delta^{1-2/p} \|\widehat{\phi_N}(\xi)\|_{L^q(|\xi| > \delta R_0)} \end{aligned}$$

Now

$$\begin{aligned}
\|\widehat{\phi}_N(\xi)\|_{L^q(|\xi|>\delta R_0)} &\leq \left\| \frac{(\xi_1 + i\xi_2)^m}{|\xi|^m} \widehat{\phi}_N(\xi) \right\|_{L^q(|\xi|>\delta R_0)} \\
&\leq (\delta R_0)^{-m} \left\| \sum_{|\alpha|=m} \frac{m!}{\alpha!} \widehat{D^\alpha \phi_N}(\xi) \right\|_{L^q} \leq (\delta R_0)^{-m} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \|D^\alpha \phi_N\|_{L^{q'}} \\
&\leq (\delta R_0)^{-m} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (C_* N)^m \leq (\delta R_0)^{-m} (2C_* N)^m
\end{aligned}$$

for $m \leq N$ from where (d) follows. \square

Now we combine the above estimates to obtain the precise decay for the solutions to the linear equation.

Proposition 5.6. *Assume that $\mu \in W^{\alpha,2}(\mathbb{C})$ is a Beltrami coefficient, with compact support inside of \mathbb{D} , such that $\|\mu\|_\infty \leq \kappa$ and $\|\mu\|_{W^{\alpha,2}(\mathbb{C})} \leq \Gamma_0$. For each $\lambda \in \partial\mathbb{D}$ and each $k \in \mathbb{C}$, let $\psi = \psi_\lambda(z, k)$ be the quasiconformal mapping satisfying*

$$\bar{\partial}\psi_\lambda(z, k) = \frac{\bar{k}}{k} \lambda e_{-k}(z) \mu(z) \partial\psi_\lambda(z, k) \quad (5.12)$$

and normalized by

$$\psi_\lambda(z, k) - z = \mathcal{O}(1/z), \quad z \rightarrow \infty.$$

There exists positive constants $C = C(\kappa)$ and $b = b(\kappa)$ such that

$$|\psi_\lambda(z, k) - z| \leq \frac{C \Gamma_0}{|k|^{b\alpha}}$$

for every $z, k \in \mathbb{C}$ and every $\lambda \in \partial\mathbb{D}$.

Proof. Let $b > 0$ a constant to be defined, and let $n_0 \in \mathbb{N}$. As in [15], we can represent

$$\begin{aligned}
\psi_\lambda(z, k) - z &= C \int_{\mathbb{D}} \frac{\bar{\partial}\psi_\lambda(w, k)}{w - z} dA(w) \\
&= C \int_{\mathbb{C}} \Phi(w, z) (g(w, k) + h(w, k)) dA(w)
\end{aligned}$$

with $g = g_\lambda(z, k)$ and $h = h_\lambda(z, k)$ as in Lemma 5.1.

Recall that we have control on \hat{g} for low frequencies by property 3 in Lemma 5.1, whereas h will be controlled by the ellipticity. It is also convenient to consider the mollification $\Phi_{\delta, N}$ of Φ given in lemma 5.5 for N to be chosen along the proof. We will therefore estimate the following four terms separately. The first three are dealt with by the usual ellipticity theory and

the Sobolev regularity of the Cauchy Kernel. Hence the estimates will depend on a suitable exponent $s = s(\kappa)$. It is in the last term where α, p will appear. Then we will chose the exponent $p = 4/3$ which will yield better constants.

$$\mathbf{I} = \int_{\mathbb{D}} \Phi(w, z) h(w) dA(w),$$

$$\mathbf{II} = \int_{\mathbb{D}} (\Phi(w, z) - \Phi_{\delta, N}(w, z)) g dA(w),$$

$$\mathbf{III} = \int_{|\xi| < R} \widehat{\Phi_{\delta, N}}(\xi, z) \widehat{g}(\xi, k) dA(\xi),$$

$$\mathbf{IV} = \int_{|\xi| > R} \widehat{\Phi_{\delta, N}}(\xi, z) \widehat{g}(\xi, k) dA(\xi)$$

I: The tail Fix $s = s(\kappa)$ such that $\kappa \|T\|_s < 1$. Then we have

$$\begin{aligned} \left| \int_{\mathbb{D}} \Phi(w, z) h(w) dA(w) \right| &\leq \|\Phi(\cdot, z)\|_{L^{\frac{s}{s-1}}(\mathbb{D})} \|h\|_{L^s(\mathbb{D})} \\ &\leq C(\kappa, s) (\kappa \|T\|_s)^{n_0} \end{aligned}$$

since by Lemma 5.5 (a), the norm $\|\Phi(\cdot, z)\|_{L^{\frac{s}{s-1}}(\mathbb{D})}$ does not depend on z . Take now,

$$n_0 \geq C(\kappa, s) + b \frac{\log(|k|)}{-\log(\kappa \|T\|_s)} = C(\kappa)(1 + b \log(|k|)) \quad (5.13)$$

so that,

$$C(\kappa, s) (\kappa \|T\|_s)^{n_0} \leq |k|^{-b} \quad (5.14)$$

and hence

$$|\mathbf{I}| \leq |k|^{-b}.$$

II: The error of mollification. We will use Lemma 5.5 with exponent $1 < s' < 2$ with $\frac{1}{s} + \frac{1}{s'} = 1$. Thus $0 < \epsilon < 1 - \frac{2}{s}$ and $0 < \delta < |k|^{\frac{-b}{\epsilon}}$. Then it follows from Lemma 5.5 (c) and Lemma 5.1 that

$$|\mathbf{II}| \leq \|g\|_{L^s(\mathbb{D})} \|\Phi(\cdot, z) - \Phi_{\delta, N}(\cdot, z)\|_{L^{\frac{s}{s-1}}(\mathbb{D})} \leq C(\kappa, s, \epsilon) \delta^\epsilon \leq |k|^{-b}.$$

III: The mollification at high frequencies . We now choose the optimal value of N . We use Plancherel's Theorem and Lemma 5.5 (e) with $m = N$ (assuming $R\delta > 1$), to get

$$\begin{aligned} \left| \int_{|\xi| \geq R} \widehat{\Phi_{\delta,N}}(\xi, z) \widehat{g}(\xi, k) dA(\xi) \right| &\leq \|g\|_{L^2(\mathbb{C})} \|\widehat{\Phi_{\delta,N}}(\cdot, z)\|_{L^2(|\xi| \geq R)} \\ &\leq C(s, \kappa) (C_* N)^N \delta^{\frac{2}{s}-1} (\delta R)^{-N} \leq |k|^{-b}. \end{aligned} \quad (5.15)$$

Let us plug in the value of δ and choose N to obtain the optimal value of R . Namely first $\delta^{\frac{2}{s}-1} \approx |k|^{2b}$. Thus we obtain that

$$R^N \geq (CN)^N |k|^{3b + \frac{Nb}{\epsilon}}$$

or

$$R \geq C |k|^{\frac{b}{\epsilon}} N |k|^{\frac{3b}{N}}.$$

With the optimal $N = [3b \log(k)] + 1$ we get the condition

$$R \geq C |k|^{\frac{b}{\epsilon}} \log(|k|).$$

Imposing $b < \epsilon$ we obtain that for large $|k|$ it is enough to take

$$R \geq \frac{|k|}{4}. \quad (5.16)$$

IV: The mollification at low frequencies. The final term is the crucial one. Take $1 < p < 2$, and $q = \frac{p}{p-1}$. Then

$$\left| \int_{|\xi| < R} \widehat{g}(\xi, k) \widehat{\Phi_{\delta,N}}(\xi, z) dA(w) \right| \leq \left(\int_{|\xi| < R} |\widehat{g}(\xi, k)|^q dA(\xi) \right)^{\frac{1}{q}} \|\widehat{\Phi_{\delta,N}}(\cdot, z)\|_{L^p(|\xi| < R)}$$

For $|k| \geq 2R$ we can use Lemma 5.1 and obtain

$$\left(\int_{|\xi| < R} |\widehat{g}(\xi, k)|^q dA(\xi) \right)^{\frac{1}{q}} \leq C(\alpha, \kappa, p) M(p)^{n_0} \frac{\Gamma_0}{|k|^\alpha}$$

At the same time, the other factor is bounded with the help of Lemma 5.5 (d), which is allowed since $p < 2$. More precisely, we have

$$\begin{aligned} \|\widehat{\Phi_{\delta,N}}(\cdot, z)\|_{L^p(|\xi| < R)} &= \left(\int_{|\xi| < R} |\widehat{\Phi_{\delta,N}}(\xi, z)|^p dA(\xi) \right)^{\frac{1}{p}} \\ &\leq C(p) R^{\frac{2}{p}-1} \left(\int_{|\xi| < R} |\widehat{\Phi_{\delta,N}}(\xi, z)|^2 dA(\xi) \right)^{\frac{1}{2}} \\ &\leq C(p) R^{\frac{2}{p}-1} \|\Phi_{\delta,N}(\cdot, z)\|_{L^2(\mathbb{C})} \\ &\leq C(p) \left(\frac{R}{\delta} \right)^{\frac{2}{p}-1} \leq |k|^{\left(\frac{2b}{\epsilon}\right)\left(\frac{2}{p}-1\right)} \end{aligned}$$

Here we have inserted the values of R and δ from **II** and **III**. Thus, whenever $|k| \geq 2R \geq \frac{|k|}{2}$ we have

$$\left| \int_{|\xi| < R} \widehat{g}(\xi, k) \widehat{\Phi_{\delta, N}}(\xi, z) dA(w) \right| \leq C(\alpha, \kappa, p) \frac{\Gamma_0}{|k|^\alpha} M(p)^{n_0} |k|^{\left(\frac{2b}{\epsilon}\right)\left(\frac{2}{p}-1\right)}$$

Now since $\|T\|_{L^p} \leq C(p-1)$ it follows that the best choice is $p = 4/3$. Inserting this and the value of n_0 from (5.13) in the previous equation,

$$C(\alpha, \kappa, p) \Gamma_0 \frac{1}{|k|^\alpha} |k|^{C(\kappa)b} |k|^{\frac{b}{\epsilon}} \leq C(\alpha, \kappa, p) \Gamma_0 |k|^{bC(\kappa)\epsilon^{-1}-\alpha}$$

Finally we want that **(IV)** is controlled by k^{-b} as well. Since $\epsilon = \epsilon(\kappa) < 1$ and we already asked $b < \epsilon$, we end up getting that it suffices that

$$b < \min \left\{ \frac{\epsilon\alpha}{C}, \epsilon \right\} = \frac{\epsilon\alpha}{C}.$$

Here $C = C(\kappa) > 1$ and we have use that $\alpha < 1$. The proof is concluded. \square

5.2 Estimates for the nonlinear equation

Now that the behavior at $k \rightarrow \infty$ of the solutions to the linearized equation (5.12) is known, it is time to study the behavior of the complex geometric optics solutions.

Theorem 5.7. *Let $\mu \in W^{\alpha,2}(\mathbb{C})$ be a Beltrami coefficient, real valued, compactly supported in $\frac{1}{4}\mathbb{D}$, such that $\|\mu\|_\infty \leq \frac{K-1}{K+1}$ and $\|\mu\|_{W^{\alpha,2}(\mathbb{C})} \leq \Gamma_0$. Let $\phi = \phi_\lambda(z, k)$ be the solution to*

$$\begin{cases} \bar{\partial}\phi_\lambda(z, k) = -\frac{\bar{k}}{k} \lambda \mu(z) e_{-k}(\phi_\lambda(z, k)) \overline{\partial\phi_\lambda(z, k)} \\ \phi_\lambda(z, k) - z = \mathcal{O}(1/|z|) \text{ as } |z| \rightarrow \infty. \end{cases}$$

There exists constants $C = C(K) > 0$ and $b = b(K)$ such that

$$|\phi_\lambda(z, k) - z| \leq \frac{C \Gamma_0^{\frac{1}{K}}}{|k|^{b\alpha}}$$

for every $z \in \mathbb{C}$, $k \in \mathbb{C}$ and $\lambda \in \partial\mathbb{D}$.

Proof. Since the estimate we look for is uniform in z and λ , it suffices to show equivalent decay for the inverse mapping $\psi_\lambda = \phi_\lambda^{-1}$. But ψ_λ is the only quasiconformal mapping on the plane that satisfies both the equation

$$\bar{\partial}\psi_\lambda(z, k) = \frac{\bar{k}}{k} \lambda e_{-k}(z) \mu(\psi_\lambda(z, k)) \partial\psi_\lambda(z, k)$$

(compare with (5.12)) and the condition $\psi_\lambda(z, k) - z = \mathcal{O}(1/|z|)$ as $|z| \rightarrow \infty$. Then, we just need to show that the coefficient

$$\mu(\psi_\lambda(z, k))$$

satisfies the assumptions of Proposition 5.6. First, it is obvious that

$$\|\mu \circ \psi_\lambda(\cdot, k)\|_\infty \leq \frac{K-1}{K+1}$$

and it is also obvious that $\text{supp}(\mu \circ \psi_\lambda(\cdot, k)) \subset \mathbb{D}$ (this follows from Koebe's $\frac{1}{4}$ Theorem). Then, it remains to prove that $\mu \circ \psi_\lambda \in W^{\beta,2}(\mathbb{C})$ for some $\beta \in (0,1)$. But this follows from Proposition 4.2. Indeed, since $\mu \in W^{\alpha,2}(\mathbb{C}) \cap L^\infty(\mathbb{C})$, we have $\mu \circ \psi_\lambda \in W^{\beta,2}(\mathbb{C})$ with

$$\|\mu \circ \psi_\lambda\|_{W^{\beta,2}(\mathbb{C})} \leq C \Gamma_0^{\frac{1}{K}}$$

for any $0 < \beta < \frac{\alpha}{K}$, where $C = C(\alpha, \beta, K)$. Note also that β behaves linearly as a function of α , with constant depending only on K . So the result follows. \square

Remark 5.8. In the above result, the assumption $\text{supp}(\mu) \in \frac{1}{4}\mathbb{D}$ is not restrictive. Indeed, if $\text{supp}(\mu) \subset D(0, R)$ for some $R > 0$ then the function $\mu_R(z) = \mu(4Rz)$ defines a new Beltrami coefficient, compactly supported in $\frac{1}{4}\mathbb{D}$, does not change the ellipticity bound, and

$$\|D^\alpha \mu_R\|_{L^2(\mathbb{C})} = (4R)^{1-\alpha} \|D^\alpha \mu\|_{L^2(\mathbb{C})}.$$

One can then apply the previous Theorem to this coefficient μ_R and obtain estimates for the complex geometric optics solutions. But $f_{\mu_R}(z, k) = f_\mu(4Rz, \frac{k}{4R})$ and in fact if we represent these solutions as $f_\mu(z, k) = \exp(ik\phi_\mu(z, k))$, then

$$\phi_{\mu_R}(z, k) = \frac{1}{4R} \phi_\mu\left(4Rz, \frac{k}{4R}\right).$$

so the estimates for ϕ_{μ_R} coming from the previous theorem give similar estimates for ϕ_μ , modulo a power of R .

Now as discovered in [11] the unimodular complex parameter λ allows to push the decay estimates to complex geometric optics solutions to the γ -harmonic equation. As always, given a real Beltrami coefficient ν we denote by $f_\nu(z, k) = e^{ikz} M_\nu(z, k)$ the complex geometric optics solutions to $\bar{\partial}f = \nu \bar{\partial}f$.

Theorem 5.9. *Let μ be as in Theorem 5.7, and define*

$$u = \text{Re}(f_\mu) + i \text{Im}(f_{-\mu}).$$

There exist a function $\epsilon = \epsilon(z, k)$ and positive constants $C = C(K)$ and $b = b(K)$ such that

$$(a) \quad u(z, k) = e^{ik(z + \epsilon(z, k))}.$$

$$(b) \quad |\epsilon(z, k)| \leq \frac{C \Gamma_0^{\frac{1}{K}}}{|k|^{b\alpha}} \text{ for each } z, k \in \mathbb{C}.$$

Further, a similar estimate holds for $\tilde{u} = \operatorname{Re}(f_{-\mu}) + i \operatorname{Im}(f_{\mu})$.

Proof. A calculation shows that u may be rewritten as

$$u = f_{\mu} \frac{1 + \frac{\overline{f_{\mu}} - \overline{f_{-\mu}}}{f_{\mu} + f_{-\mu}}}{1 + \frac{f_{\mu} - f_{-\mu}}{\overline{f_{\mu}} + \overline{f_{-\mu}}}}.$$

Thus, the Theorem will follow if we find a function $\epsilon(z, k)$ such that

$$|\epsilon(z, k)| \leq \frac{C \Gamma_0^{\frac{1}{K}}}{|k|^{b\alpha}} \text{ and}$$

$$\left| \frac{f_{\mu} - f_{-\mu}}{f_{\mu} + f_{-\mu}} \right| \leq 1 - e^{|k \epsilon(z, k)|}$$

Following [11, Lemma 8.2], it suffices to see that

$$\inf_t \left| \frac{f_{\mu} - f_{-\mu}}{f_{\mu} + f_{-\mu}} + e^{it} \right| \geq e^{|k \epsilon(z, k)|}$$

For this, define $\Phi_t(z, k) = e^{-\frac{it}{2}} (f_{\mu} \cos(t/2) + i f_{-\mu} \sin(t/2))$. It follows easily that for each fixed k ,

$$\begin{cases} |e^{-ikz} \Phi_t(z, k) - 1| = \mathcal{O}(1/z) & \text{as } |z| \rightarrow \infty \\ \bar{\partial} \Phi_t = e^{-it} \mu \bar{\partial} \Phi_t \end{cases}$$

Thus, by uniqueness in Theorem 2.2, Φ_t is nothing but the complex geometric optics solution $\Phi_t = f_{\lambda\mu}$ with $\lambda = e^{-it}$. But then

$$\frac{f_{\mu} - f_{-\mu}}{f_{\mu} + f_{-\mu}} + e^{it} = \frac{2 e^{it} \Phi_t}{f_{\mu} + f_{-\mu}} = \frac{f_{\lambda\mu}}{f_{\mu}} \frac{2 e^{it}}{1 + \frac{M_{-\mu}}{M_{\mu}}}$$

On the other hand, from Theorem 5.7 we get that

$$e^{-|k \epsilon(z, k)|} \leq |M_{\mu}(z, k)| = |e^{ik(\phi_{\mu}(z, k) - z)}| \leq e^{|k \epsilon(z, k)|}$$

where $|\epsilon(z, k)| \leq \frac{C \Gamma_0^{\frac{1}{K}}}{|k|^{b\alpha}}$ and

$$e^{-2|k \epsilon(z, k)|} \leq \frac{|f_{\lambda\mu}(z, k)|}{|f_{\mu}(z, k)|} \leq e^{2|k \epsilon(z, k)|}$$

uniformly for $\lambda \in \partial\mathbb{D}$. Finally, by Theorem 2.2, we also have $\operatorname{Re} \left(\frac{M_{-\mu}}{M_{\mu}} \right) > 0$, so that the result follows. \square

6 Proof of Theorem 1.1

In order to get stability from Λ_γ to μ , we will need the stability result for the complex geometric optics solutions in terms of a Sobolev norm. It comes as an interpolating consequence of the L^∞ stability result given at Theorem 5.9 and of the regularity of the solutions to a Beltrami equation with Sobolev coefficients (see Theorem 4.6).

Theorem 6.1. *Let μ_1, μ_2 be Beltrami coefficients, compactly supported in \mathbb{D} , such that $\|\mu_j\| \leq \frac{K-1}{K+1}$ and $\|\mu_j\|_{W^{\alpha,2}(\mathbb{C})} \leq \Gamma_0$. Let f_{μ_j} denote the complex geometric optics solutions to $\bar{\partial} f_{\mu_j} = \mu_j \bar{\partial} f_{\mu_j}$. Then, for each $\theta \in (0, \frac{1}{K})$ we have*

$$\|f_{\mu_1} - f_{\mu_2}\|_{\dot{W}^{1,(1+\alpha\theta)2}(\mathbb{D})} \leq C (1 + \Gamma_0)^{\frac{1}{K}} \left| \log \frac{1}{\rho} \right|^{-b\alpha^2}$$

for some constants $C = C(|k|, K) > 0$ and $b = b(K) > 0$. In particular, the same bound holds with the $\dot{W}^{1,2}(\mathbb{D})$ -norm.

Proof. The subexponential growth obtained in Theorem 5.9 entitled us to apply Theorem 2.4 **B** to the solutions u_{γ_i} . Since they are equivalent to the corresponding f_μ we achieve the estimate

$$\|f_{\mu_1}(\cdot, k) - f_{\mu_2}(\cdot, k)\|_{L^\infty(\mathbb{D})} \leq \frac{C \Gamma_0^{\frac{1}{K^2}}}{|\log(\rho)|^{b\alpha}}$$

for some positive constants $C = C(k, K)$ and $b = b(K)$. On the other hand, from Theorem 4.6, for every $\theta \in (0, \frac{1}{K})$ we have

$$\begin{aligned} \|f_{\mu_1}(\cdot, k) - f_{\mu_2}(\cdot, k)\|_{\dot{W}^{1+\alpha\theta,2}(\mathbb{D})} &= \|D^{1+\theta\alpha}(f_{\mu_1}(\cdot, k) - f_{\mu_2}(\cdot, k))\|_{L^2(\mathbb{D})} \\ &\leq C e^{C|k|} (1 + |k|) (\Gamma_0 + |k|^\alpha)^\theta. \end{aligned}$$

As in Theorem 4.6, here C may depend on K . Let $\varphi \in \mathcal{C}^\infty(\mathbb{C})$ be a cut-off function, compactly supported in $2\mathbb{D}$, such that $\varphi|_{\mathbb{D}} = \chi_{\mathbb{D}}$. Then the above estimates imply that

$$\begin{aligned} \|(f_{\mu_1}(\cdot, k) - f_{\mu_2}(\cdot, k))\varphi\|_{L^\infty(\mathbb{C})} &\leq \frac{C \Gamma_0^{\frac{1}{K^2}}}{|\log(\rho)|^{b\alpha}} \\ \|(f_{\mu_1}(\cdot, k) - f_{\mu_2}(\cdot, k))\varphi\|_{\dot{W}^{1+\alpha\theta,2}(\mathbb{C})} &\leq C e^{C|k|} (1 + |k|) (\Gamma_0 + |k|^\alpha)^\theta \end{aligned}$$

where, as usually, $\|\cdot\|_{\dot{W}^{s,p}(\mathbb{C})}$ denotes the homogeneous Sobolev norm. Now, an interpolation argument shows that for each $0 < \beta < 1$ we have

$$\begin{aligned} \|\varphi(f_{\mu_1}(\cdot, k) - f_{\mu_2}(\cdot, k))\|_{\dot{W}^{(1+\alpha\theta)\beta, \frac{2}{\beta}}(\mathbb{C})} &\leq C e^{C\beta|k|} (1 + |k|)^\beta (\Gamma_0 + |k|^\alpha)^{\beta\theta} \frac{\Gamma_0^{\frac{1-\beta}{K^2}}}{|\log(\rho)|^{b\alpha(1-\beta)}} \\ &\leq C e^{C\beta|k|} (1 + |k|)^{\beta(1+\alpha\theta)} (1 + \Gamma_0)^{\frac{1}{K^2} + \beta(\theta - \frac{1}{K^2})} \frac{1}{|\log(\rho)|^{b\alpha(1-\beta)}} \end{aligned}$$

where $C = C(K)$. In particular, for $\beta = \frac{1}{1+\alpha\theta}$ we get that

$$\begin{aligned} \|f_{\mu_1} - f_{\mu_2}\|_{\dot{W}^{1,(1+\alpha\theta)2}(\mathbb{D})} &\leq \|(f_{\mu_1} - f_{\mu_2}) \varphi\|_{W^{1,(1+\alpha\theta)2}(\mathbb{C})} \\ &\leq C e^{C|k|} (1 + |k|) (1 + \Gamma_0)^{\frac{1}{K^2} + \frac{1}{1+\alpha\theta}(\theta - \frac{1}{K^2})} \frac{1}{|\log(\rho)|^{\frac{b\alpha^2\theta}{1+\alpha\theta}}} \end{aligned}$$

Here the sharp modulus of continuity is obtained when the logarithm has the bigger exponent, which is given for $\theta = \frac{1}{K}$. Thus, we end up with

$$\|f_{\mu_1} - f_{\mu_2}\|_{\dot{W}^{1,(1+\alpha\theta)2}(\mathbb{D})} \leq C(|k|) (1 + \Gamma_0)^{\frac{1}{K}} \frac{1}{|\log(\rho)|^{\frac{b}{K}\alpha^2}}$$

as claimed. \square

It just remains to see how the previous estimate drives us to the final stability bounds for the Beltrami coefficients (and therefore for the conductivities). To do this, the following interpolation Lemma will be needed. Note that it includes L^p spaces with $p < 1$.

Lemma 6.2 (Interpolation). *Let $0 < p_0 \leq 2$ and $2 < p_1 \leq \infty$. Let θ be such that*

$$\frac{1}{2} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}.$$

Then

$$\|f\|_{L^2} \leq \|f\|_{L^{p_0}}^\theta \|f\|_{L^{p_1}}^{1-\theta}$$

for any $f \in L^{p_0} \cap L^{p_1}$.

Proof. The proof is adapted for the usual Riesz method for interpolation with a little extra care when $p_0 < 1$. We choose $r < p_0$ and define exponents q_0, q_1, q_2 such that

$$\frac{1}{r} = \frac{1}{p_0} + \frac{1}{q_0} \quad \frac{1}{r} = \frac{1}{2} + \frac{1}{q_2} \quad \frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1}$$

For $z = x + iy$ in the strip $0 \leq y \leq 1$, we define the analytic function

$$G(z) = |g|^{q_2(\frac{z}{q_0} + \frac{1-z}{q_1})} \frac{g}{|g|}.$$

Notice that $|G(iy)|^{q_1} = |g|^{q_2}$, $|G(1 + iy)|^{q_0} = |g|^{q_2}$, and $|G(\theta + iy)| = |g|$. Now we introduce the function

$$I(z) = \left(\int |f|^r |G(z)|^r \right)^{\frac{1}{r}}$$

We can estimate its values at the boundary of the strip,

$$|I(iy)| \leq \|f\|_{L^{p_1}} \|G(iy)\|_{L^{q_1}} = \|f\|_{L^{p_1}} \left(\int |g|^{q_2} \right)^{\frac{1}{q_1}},$$

$$|I(1+iy)| \leq \|f\|_{L^{p_0}} \|G(iy)\|_{L^{q_0}} = \|f\|_{L^{p_0}} \left(\int |g|^{q_2} \right)^{\frac{1}{q_0}}.$$

Then we apply Phragmen-Lindelöf theorem to the function $I(z)$ obtaining that

$$I(\theta + iy) \leq \left(\|f\|_{L^{p_1}} \left(\int |g|^{q_2} \right)^{\frac{1}{q_1}} \right)^{1-\theta} \left(\|f\|_{L^{p_0}} \left(\int |g|^{q_2} \right)^{\frac{1}{q_0}} \right)^{\theta}$$

$$\leq \|g\|_{L^{q_2}} \|f\|_{L^{p_0}}^{\theta} \|f\|_{L^{p_1}}^{1-\theta}$$

But $I(\theta + iy) = \|fg\|_{L^r}$, so the result follows. \square

We are finally led to obtain the desired stability in L^2 norm of the Beltrami coefficients.

Corollary 6.3 (Proof of Theorem 1.1). *Let μ_1, μ_2 be Beltrami coefficients, compactly supported in \mathbb{D} , such that $\|\mu_j\| \leq \frac{K-1}{K+1}$ and $\|\mu_j\|_{W^{\alpha,2}(\mathbb{C})} \leq \Gamma_0$. There exists constants $b = b(K) > 0$ and $C = C(\alpha, K) > 0$ such that*

$$\|\mu_1 - \mu_2\|_{L^2(\mathbb{D})} \leq C (1 + \Gamma_0)^{\frac{1}{K^2}} \left| \log \frac{1}{\rho} \right|^{-b\alpha^2}$$

where $\rho = \|\Lambda_1 - \Lambda_2\|_{H^{\frac{1}{2}}(\partial\mathbb{D}) \rightarrow H^{-\frac{1}{2}}(\partial\mathbb{D})}$.

Proof. Denote by f_i the complex geometric optics solution f_{μ_i} of $\bar{\partial}f = \mu_i \bar{\partial}f$ with $k = 1$. Then,

$$|\mu_1 - \mu_2| = \left| \frac{\bar{\partial}f_1 \bar{\partial}f_2 - \bar{\partial}f_2 \bar{\partial}f_1}{\bar{\partial}f_1 \bar{\partial}f_2} \right| = \left| \frac{-\bar{\partial}f_1 (\bar{\partial}f_1 - \bar{\partial}f_2) + (\bar{\partial}f_1 - \bar{\partial}f_2) \bar{\partial}f_1}{\bar{\partial}f_1 \bar{\partial}f_2} \right|$$

$$\leq \frac{|\bar{\partial}f_1 - \bar{\partial}f_2|}{|\bar{\partial}f_2|} + |\mu_1| \frac{|\partial f_1 - \partial f_2|}{|\partial f_2|} \leq 2 \frac{|Df_1 - Df_2|}{|\partial f_2|}$$

because $|Df_j| = |\partial f_j| + |\bar{\partial}f_j|$. Therefore, for any $s > 0$

$$\|\mu_1 - \mu_2\|_{L^s(\mathbb{D})} \leq 2 \left\| \frac{Df_1 - Df_2}{\bar{\partial}f_2} \right\|_{L^s(\mathbb{D})}.$$

Now, let $p \in (0, \frac{2}{K-1})$ and $\theta \in (0, 1/K)$. Then put $\frac{1}{s} = \frac{1}{2(1+\alpha\theta)} + \frac{1}{p}$. An application of Hölder's inequality gives us that

$$\left\| \frac{Df_1 - Df_2}{\bar{\partial}f_2} \right\|_{L^s(\mathbb{D})} \leq C(\alpha, \theta) \|Df_1 - Df_2\|_{L^{2(1+\alpha\theta)}(\mathbb{D})} \left\| \frac{1}{\bar{\partial}f_2} \right\|_{L^p(\mathbb{D})}$$

$$\leq C(\alpha, \theta) \|f_1 - f_2\|_{\dot{W}^{1,2(1+\alpha\theta)}(\mathbb{D})} \left\| \frac{1}{\bar{\partial}f_2} \right\|_{L^p(\mathbb{D})}.$$

Now, using Lemma 4.7 and Theorem 6.1, we obtain the estimate

$$\|\mu_1 - \mu_2\|_{L^s(\mathbb{D})} \leq C (1 + \Gamma_0)^{\frac{1}{K}} \left| \log \frac{1}{\rho} \right|^{-b\alpha^2}$$

where $C > 0$ depends on α, θ , and K , and $b > 0$ depends on K . Finally, if $s \geq 2$ then we are done, since μ_i are compactly supported in \mathbb{D} . But in general we only know $0 < \frac{2}{K} < s$ so one could well have $s < 1$. In this case, in order to get L^2 estimates only interpolation between L^s and L^∞ is needed, as in Lemma 6.2. Namely,

$$\|\mu_1 - \mu_2\|_{L^2(\mathbb{D})} \leq \|\mu_1 - \mu_2\|_{L^s(\mathbb{D})}^{\frac{s}{2}} \|\mu_1 - \mu_2\|_{L^\infty(\mathbb{D})}^{\frac{2-s}{2}}$$

and now the stability estimate looks like

$$\|\mu_1 - \mu_2\|_{L^2(\mathbb{D})} \leq C (1 + \Gamma_0)^{\frac{1}{K^2}} \left| \log \frac{1}{\rho} \right|^{-b\alpha^2}$$

where the constants may have changed. \square

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